1. Introduction to Differential Models

Differential equations is one of the most interesting topics in Differential Calculus; they concern with the problem to find a regular function y which satisfies a given relation, involving y itself and its derivatives, and some initial data.

The simplest example for this kind of problems is the free fall of an object

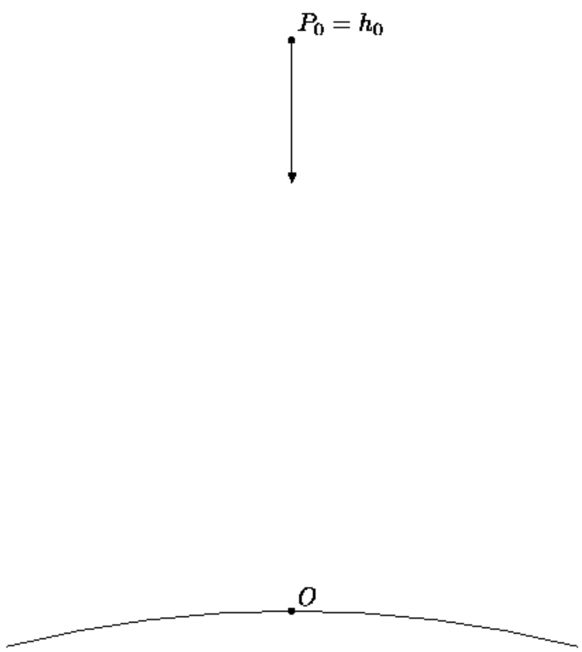


Figure 1.1: Un punto materiale soggetto alla gravità

Let us consider a material point P of mass m which falls under the sole influence of gravity and let us ignore air resistance; we have

$F = mg$.

Experimental evidence shows that P falls down; we can use as reference system the straight line traveled by the material point assuming that the origin coincides with the ground and considering the line oriented upward.

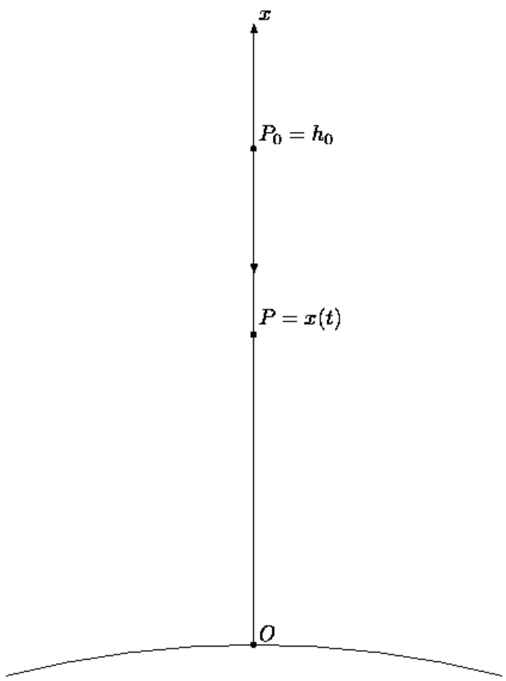


Figure 1.2: Il sistema di riferimento

The velocity of P is directed downward and, if $x(t)$ is the position of the material point P at time t, P moves down with velocity

$$v(t) = \dot{x}(t)$$

and acceleration

$$a(t) = \ddot{x}(t)$$

Since P falls under the sole influence of gravity, only the force $F = mg$ acts on P and, by the first Newton Law we have

$$ma(t) = -mg$$

e quindi

$$\ddot{x}(t) = g \tag{1.1}$$

We can find $x(t)$ integrating twice with respect to t and taking into account that $t_0 = 0$ and $x(t_0) = h$.

We have

$$\dot{x}(t) = -gt + c_1 \tag{1.2}$$

and

$$x(t) = -\frac{1}{2}gt^2 + c_1 t + c_0 \tag{1.3}$$

In order to determine uniquely the motion we should give values for c_0 e c_1.

We can do this using initial values of velocity and position of the material point P

In fact, we can easily show, using 1.2 and 1.3, that:

$$v_0 = \dot{x}(0) = c_1 \qquad h_0 = x(0) = c_0 \tag{1.4}$$

and this seems natural as to determine the motion of the material point P is necessary to know also its initial position and velocity.

If we take account of these data we can say that P moves along the x axis according to the law

$$x(t) = -\frac{1}{2}gt^2 + v_0 t + h_0 \tag{1.5}$$

Equations for a falling body can also be obtained using the Conservation of Energy Principle

At time t the potential energy of P is

$$U(t) = mgx(t)$$

while its kinetic energy is

$$\frac{1}{2}m\dot{x}^2(t)$$

and its total energy

$$E(t) = \frac{1}{2}m\dot{x}^2(t) + mgx(t)$$

remains constant during the motion.

$$\frac{1}{2}m\dot{x}^2(t) + mgx(t) = mk \tag{1.6}$$

Using initial data v_0 ed h_0 we can compute

$$k = \frac{1}{2}mv_0^2 + mgh_0$$

The resulting differential model

$$\begin{cases} \frac{1}{2}m\dot{x}^2(t) + mgx(t) = mk \\ \frac{1}{2}mv_0^2 + mgh_0 = k \end{cases}$$

can be used to describe the position $x(t)$ of the material point P , however it is more difficult to derive x explicitly; moreover there is a problem about uniqueness of the solution.

1.6 can be written as

$$\frac{1}{2}\dot{x}^2(t) = k - gx(t) \tag{1.7}$$

and this equality shows that

$k - gx(t)$ must be positive, so that $x(t) \leq \frac{k}{g}$.

We also remark that

$x(t) = \frac{k}{g}$ is a contant solution of equation 1.7

In order to find non-constant solutions we can proceed as follows: take the square root of both sides

$$\dot{x}(t) = \pm\sqrt{2k - 2gx(t)} \tag{1.8}$$

divide by the second member

$$\frac{\dot{x}(t)}{\sqrt{2k - 2gx(t)}} = \pm 1 \tag{1.9}$$

multiply both sides by g

$$\frac{g\dot{x}(t)}{\sqrt{2k - 2gx(t)}} = \pm g \tag{1.10}$$

integrate from $t_0 = 0$ to t,

$$\int_0^t \frac{g\dot{x}(s)}{\sqrt{2k - 2gx(s)}} ds = \pm gt \tag{1.11}$$

Now we can integrate by substitution: we let

$$u = x(s) \qquad , \qquad du = \dot{x}(s)ds$$

whence,

$$\int_{h_0}^{x(t)} \frac{gdu}{\sqrt{2k - 2gu}} = \pm gt \tag{1.12}$$

Since $v_0 = \pm\sqrt{2k - 2gh_0}$, we can assert that

$$\sqrt{2k - 2gx(t)} - \sqrt{2k - 2gh_0} = \mp gt \tag{1.13}$$

$$\sqrt{2k - 2gx(t)} = \mp gt \pm v_0 \tag{1.14}$$

$$2k - 2gx(t) = (\mp gt \pm v_0)^2 \tag{1.15}$$

$$x(t) = \frac{k}{g} - \frac{1}{2g}(\mp gt \pm v_0)^2 \tag{1.16}$$

In doing calculations we kept the signs of gt e di v_0 consistent, however we must observe that the choice of the sign holds true as long as $\dot{x}(t)$, i.e. velocity, doesn't vanish.

This possibility never occurs if $v_0 < 0$ while it happens at $t_0 = \frac{v_0}{g}$ if $v_0 > 0$.

In this case we have

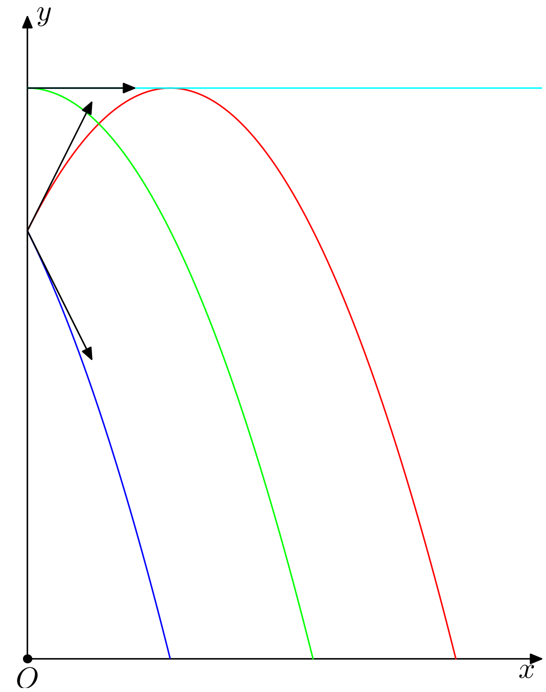
$$\dot{x}(t_0) = 0$$

and we have no criteria to choose the sign of velocity in 1.8.

However, since $\dot{x}(t) = 0$, we obtain from 1.8 that $x(t) = \frac{k}{g}$ and it must be $x(t) \leq \frac{k}{g}$, we deduce that x cannot increase so that

$$\dot{x}(t) = -\sqrt{2k - 2gx(t)} \tag{1.17}$$

Let us finally remark that, since we used the Conservation of Energy Principle, we also found a constant solution $x(t) = \frac{k}{g}$; however this solution is meaningless for the motion of the falling body.



2. Ordinary Differential Equations with Separable Variables.

Solving a Differential Equation with Separable Variables means to find a differentiable function y, such that

$$y'(x) = f(x)g(y(x))$$

for given f, g .

More precisely:

Let $I, J \subset \mathbb{R}$ be open non-empty intervals $f : I \longrightarrow \mathbb{R}$, $g : J \longrightarrow \mathbb{R}$ be functions; we solve the ordinary differential equation with separable variables

$$y'(x) = f(x)g(y(x)) \tag{2.1}$$

if we find an interval $I' \subset I$ and a function $y : I' \longrightarrow J$ such that 2.1 is satisfied for all $x \in I'$

When initial data are also given, we speak of Cauchy problem.

Let $I, J \subset \mathbb{R}$ be open intervals, let $x_0 \in I$, $y_0 \in J$, $f : I \longrightarrow \mathbb{R}$ and let $g : J \longrightarrow \mathbb{R}$ be functions.

The Cauchy problem with separable variables consists of finding an intyerval $I' \subset I$ and a function $y : I' \longrightarrow \mathbb{R}$, derivabile, such that

$$\begin{cases} y'(x) = f(x)g(y(x)) \quad , \quad \forall x \in I' \\ y(x_0) = y_0 \end{cases} \tag{2.2}$$

The following existence and uniqueness theorem holds.

Theorem 2.1 *Let $I, J \subset \mathbb{R}$, be open intervals, let $x_0 \in I$, $y_0 \in J$ and let $f : I \longrightarrow \mathbb{R}$, $g : J \longrightarrow \mathbb{R}$ be continuous functions. Let us suppose that $g(y) \neq 0$, per ogni $y \in J$*

Then there exists an interval $I' \subset I$ and a differentiable function $y : I' \longrightarrow J$ solution of the Cauchy problem 2.2. Moreover the solution in unique

In order that the theorem is true it is essential that I, J are open intervals and that $g(y) \neq 0 \ \forall y \in J$.

The latter condition is certainly true when g is continuous and $g(y_0) \neq 0$ for a suitable choiche of J).

Proof. y is a solution for the given problem iff

$$\begin{cases} \frac{y'(x)}{g(y(x))} = f(x) \\ y(x_0) = y_0 \end{cases} \tag{2.3}$$

and this is true iff

$$\int_{x_0}^{x} \frac{y'(t)}{g(y(t))} dt = \int_{x_0}^{x} f(t) dt \tag{2.4}$$

iff

$$\int_{y_0}^{y(x)} \frac{ds}{g(s)} = \int_{x_0}^{x} f(t) dt \tag{2.5}$$

iff

$$G(y(x)) - G(y_0) = F(x) - F(x_0) \tag{2.6}$$

where F e G are primitives of f and $1/g$ in I e J respectively.

By continuity assumptions we can assert that $R(G - G(y_0))$ e $R(F - F(x_0))$ are intervals and both contain 0.

Moreover 0 is an interior point for $R(G)$ because G is strictly monotone (as $g = G'$ has constant sign).

Therefore there exists an open interval I', with $x_0 \in I'$ such that the last equality holds for every $x \in \mathbb{I}'$ and we obtain

$$y(x) = G^{-1}(F(x) + G(y_0) - F(x_0)). \tag{2.7}$$

□

Following step by step the preceding proof we can find a solution of a differential equation with separable variables. However it must be remembered that, in order to divide by $g(y(x))$ it must be $g(y(x)) \neq 0$.

This fact prevent us to find solutions y such that $g(y(x)) = 0$ for some or for every x.

Particularly constants solutions are lost while, when $g(y(\bar{x})) = 0$ for some $\bar{x}$, we must carefully investigate the existence of the improper integral on the right side of the equation.

Let us now prove an a priori estimate for the solutions of a differential inequality which is very useful in many cases.

Lemma 2.1 - *Gronwall - Let $y, f : I \longrightarrow \overline{\mathbb{R}}_+$ be non-negative continuous functions, let I be an open interval, and let $c > 0$, $x_0 \in I$; The, if*

$$y(x) \leq \left| \int_{x_0}^{x} f(t) y(t) dt \right| + c$$

for every $x \in I$ we have

$$0 \le y(x) \le c e^{\left|\int_{x_0}^{x} f(t)dt\right|}$$

for everey $x \in I$.

Proof. Le us assume that $x \ge x_0$; and let us divide both sides by the right one; let us moreover multiply both sides by $f(x)$ since $f \ge 0$ and $c > 0$ the ratio on the left has non zero denominator and we can assert that

$$\frac{y(x)f(x)}{c + \int_{x_0}^{x} f(t)y(t)dt} \le f(x)$$

whence

$$\frac{d}{dx}\left[\ln\left(c + \int_{x_0}^{x} f(t)y(t)dt\right)\right] \le f(x).$$

Integrating over $[x_0, x]$ we obtain

$$\ln\left(c + \int_{x_0}^{x} f(t)y(t)dt\right) - \ln c \le \int_{x_0}^{x} f(t)dt$$
$$c + \int_{x_0}^{x} f(t)y(t)dt \le c e^{\int_{x_0}^{x} f(t)dt}.$$

and

$$y(x) \le c + \int_{x_0}^{x} f(t)y(t)dt \le c e^{\int_{x_0}^{x} f(t)dt}$$

□

Corollary 2.1 *Let $y, f : I \longrightarrow \overline{\mathbb{R}}_+$ be non-negative continuous functions, let I be an open interval and let $x_0 \in I$; If*

$$y(x) \le \left|\int_{x_0}^{x} f(t)y(t)dt\right| \qquad \forall x \in I$$

then

$$y(x) = 0 \quad \forall x \in I$$

Proof. We have

$$y(x) \le \left|\int_{x_0}^{x} f(t)y(t)dt\right| + c \qquad \forall c > 0$$

therefore

$$0 \le y(x) \le c e^{\left|\int_{x_0}^{x} f(t)dt\right|} \qquad \forall c > 0$$

and taking the limit for $c \to 0^+$, we deduce that $y(x) \equiv 0$. □

Remark. The Gronwall's Lemma is still true if we assume

$$y(x) \le \left|\int_{x_0}^{x} f(t)y(t)dt\right| + c(x)$$

where c is an increasing function in this case it yields.

$$0 \le y(x) \le c(x) e^{\left|\int_{x_0}^{x} f(t)dt\right|}$$

□

3. Some Noteworthy Differential Equations with separable variables

3.1

Let us consider the equation

$$y'(x) = y^2(x) \tag{3.1}$$

First of all let us observe that $y(x) \equiv 0$ is a constant solution.
If $y(x) \neq 0$ separate the variables

$$\frac{y'(x)}{y^2(x)} = 1 \tag{3.2}$$

integrate on $[x_0, x$

$$\int_{x_0}^{x} \frac{y'(t)}{y^2(t)} dt = x - x_0 \tag{3.3}$$

substitute $s = y(t)$ and $ds = y'(t)dt$

$$\int_{y(x_0)=y_0}^{y(x)} \frac{ds}{s^2} = x - x_0 \tag{3.4}$$

Since $\frac{1}{s^2}$ if infinite of order 2 for $s \to 0$, it is not integrable in $s = 0$ (i.e. it is not integrable even in improper sense on intervals containing 0).

As a consequence y and y_0 must have the same signand cannot be 0 .

Under this assumption we have

$$-\frac{1}{y} + \frac{1}{y_0} = x - x_0 \tag{3.5}$$

$$\frac{1}{y} = \frac{1}{y_0} + x_0 - x = c - x \tag{3.6}$$

where

$$c = \frac{1}{y_0} + x_0$$

Let us remark that the range of c is $\mathbb{R}$
All the solutions of the differential equation are

$$y(x) = \frac{1}{c-x} \tag{3.7}$$

Figure 3.1 shows their graphs.

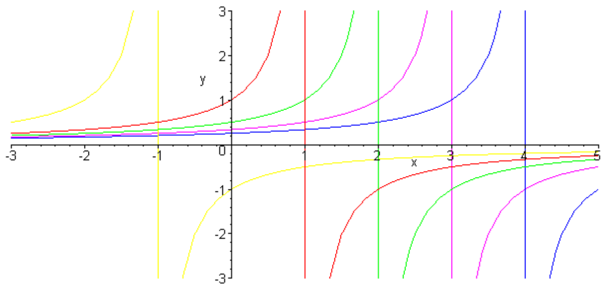


Figure 3.1:

3.2

Let us consider the equation

$$y'(x) = \sqrt{y(x)} \tag{3.8}$$

We must have $y(x) \geq 0$ and $y(x) \equiv 0$ is a constant solution.
When $y(x) \neq 0$ we can separate the variables

$$\frac{y'(x)}{\sqrt{y(x)}} = 1 \tag{3.9}$$

integrate on x_0, x

$$\int_{x_0}^{x} \frac{y'(t)}{\sqrt{y(t)}} dt = x - x_0 \tag{3.10}$$

substitute $s = y(t)$ and $ds = y'(t)dt$

$$\int_{y(x_0)=y_0}^{y(x)} \frac{ds}{\sqrt{s}} = x - x_0 \tag{3.11}$$

$\frac{1}{\sqrt{s}}$ is infinite of order $1/2$ for $s \to 0$, so that it is improperly integrable in $s = 0$ (i.e. on every interval containing 0). Therefore y and y_0 can also be 0.

We have

$$2\sqrt{y} - 2\sqrt{y_0} = x - x_0 \tag{3.12}$$

$$\sqrt{y} = \frac{1}{2}(x - x_0 + 2\sqrt{y_0}) = \frac{1}{2}(x + c) \tag{3.13}$$

where
$$c = 2\sqrt{y_0} - x_0$$

We observe that by 3.13 it must be

$$\frac{1}{2}(x+c) \geq 0 \qquad \text{cioè} \qquad x \geq -c$$

Let us also observe that c ranges over the whole $\mathbb{R}$, for suitable choice of x_0 ed y_0.

All the solutions of the differential equation are

$$y(x) = \frac{1}{4}(x+c)^2 \qquad \text{per} \qquad x \geq -c \tag{3.14}$$

Figure 3.2 shows their graphs..

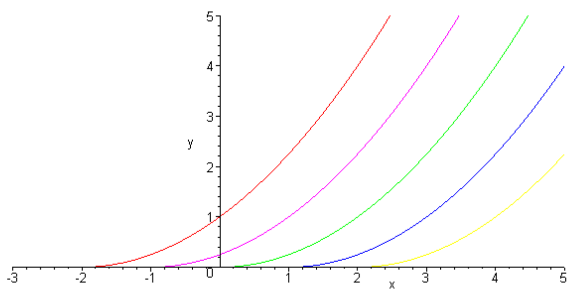


Figure 3.2:

3.3

Let us consider the equation

$$y'(x) = x\sqrt{y(x)} \tag{3.15}$$

It must be $y(x) \geq 0$ and $y(x) \equiv 0$ is a constant solution.

If $y(x) \neq 0$ we separate the variables

$$\frac{y'(x)}{\sqrt{y(x)}} = x \tag{3.16}$$

integrate on x_0, x

$$\int_{x_0}^{x} \frac{y'(t)}{\sqrt{y(t)}} dt = \int_{x_0}^{x} t dt \tag{3.17}$$

substitute $s = y(t)$ and $ds = y'(t)dt$

$$\int_{y(x_0)=y_0}^{y(x)} \frac{ds}{\sqrt{s}} = \frac{x^2}{2} - \frac{x_0^2}{2} \tag{3.18}$$

Since $\frac{1}{\sqrt{s}}$ is infinite of order $1/2$ for $s \to 0$, it is improperly integrable in $s = 0$ (i.e. on every interval containing 0).

Therefore y and y_0 can also be 0.

We have

$$2\sqrt{y} - 2\sqrt{y_0} = \frac{x^2}{2} - \frac{x_0^2}{2} \tag{3.19}$$

$$\sqrt{y} = \frac{x^2}{4} + (\sqrt{y_0} - \frac{x_0^2}{2}) = \frac{x^2}{4} + c \tag{3.20}$$

where

$$c = \sqrt{y_0} - \frac{x_0^2}{2}$$

By 3.19 it must be

$$\frac{x^2}{4} + c \geq 0 \qquad \text{cioè} \qquad \begin{cases} \text{sempre} & \text{se } c > 0 \\ |x| \geq -2c & \text{se } c < 0 \end{cases}$$

and c can be any real value.

All the solutions of the equation are

$$y(x) = \left(\frac{x^2}{4} + c\right)^2 \tag{3.21}$$

as soon as x satisfies the prescribed conditions.

Figure 3.3 shows their graphs.

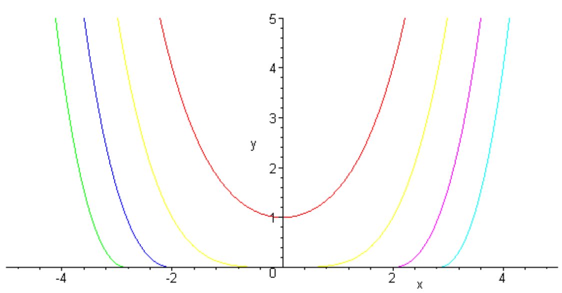


Figure 3.3:

3.4

Let us consider the equation

$$y'(x) = -x\sqrt{y(x)} \tag{3.22}$$

It must be $y(x) \geq 0$ and $y(x) \equiv 0$ is a constant solution.

If $y(x) \neq 0$ we separate the variables

$$\frac{y'(x)}{\sqrt{y(x)}} = -x \tag{3.23}$$

integrate on x_0, x

$$\int_{x_0}^{x} \frac{y'(t)}{\sqrt{y(t)}} dt = -\int_{x_0}^{x} t dt \tag{3.24}$$

substitute $s = y(t)$ and$ds = y'(t)dt$

$$\int_{y(x_0)=y_0}^{y(x)} \frac{ds}{\sqrt{s}} = -\frac{x^2}{2} + \frac{x_0^2}{2} \tag{3.25}$$

Since $\frac{1}{\sqrt{s}}$ is infinite of order $1/2$ for $s \to 0$, it is improperly integrable in $s = 0$ (i.e. on every interval containing 0). Therefore y ed y_0 can also be 0.

We have

$$2\sqrt{y} - 2\sqrt{y_0} = -\frac{x^2}{2} + \frac{x_0^2}{2} \tag{3.26}$$

$$\sqrt{y} = -\frac{x^2}{4} + (\sqrt{y_0} + \frac{x_0^2}{2}) = -\frac{x^2}{4} + c \tag{3.27}$$

where

$$c = \sqrt{y_0} + \frac{x_0^2}{2}$$

By 3.27 it must be

$$-\frac{x^2}{4} + c \geq 0 \qquad \text{cioè} \qquad \begin{cases} \text{mai} & \text{se } c < 0 \\ |x| \leq -2c & \text{se } c < 0 \end{cases}$$

and c can be only a non-negative real number.

All the solutions of the differential equation are

$$y(x) = \left(-\frac{x^2}{4} + c\right)^2 \tag{3.28}$$

under prescribed conditions on x

Figure 3.4 shows their graphs

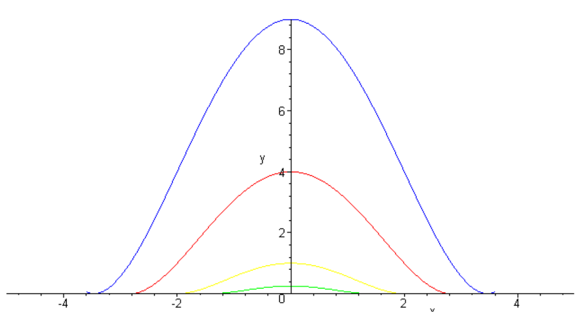


Figure 3.4:

3.5

Let us consider

$$y'(x) = \sqrt{1 - y^2(x)} \tag{3.29}$$

First of all it must be $|y(x)| \leq 1$ and $y(x) \equiv \pm 1$ are constant solutions

If $y(x) \neq \pm 1$ we separate the variables

$$\frac{y'(x)}{\sqrt{1 - y^2(x)}} = 1 \tag{3.30}$$

integrate on x_0, x

$$\int_{x_0}^{x} \frac{y'(t)}{\sqrt{1 - y^2(t)}} dt = x - x_0 \tag{3.31}$$

substitute $s = y(t)$and $ds = y'(t)dt$

$$\int_{y(x_0)=y_0}^{y(x)} \frac{ds}{\sqrt{1 - s^2}} = x - x_0 \tag{3.32}$$

Since $\frac{1}{\sqrt{1-s^2}}$ is infinite of order $1/2$ for $s \to \pm 1$, it is improperly integrable in $s = \pm 1$ (i.e. on every interval containing ± 1).

Therefore y ed y_0 can also be ± 1.

We have

$$\arcsin y(x) - \arcsin y_0 = x - x_0 \tag{3.33}$$

$$\arcsin y(x) = x - x_0 + \arcsin y_0 = x + c \tag{3.34}$$

where

$$c = \arcsin y_0 - x_0$$

By 3.34 it must be

$$|x + c| \leq \frac{\pi}{2}$$

and c can be any real value..

All the solutions of the equation are

$$y(x) = \sin(x + c) \tag{3.35}$$

assuming that x satisfies the prescribed assumptions an Figure 3.5 shows their graphs.

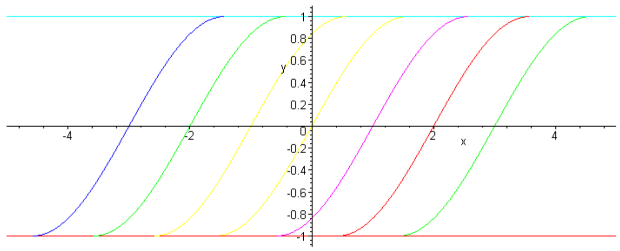


Figure 3.5:

3.6

Let us consider the Cauchy problem

$$\begin{cases} y'(x) = e^{-(y(x))^4} - 1 \\ y(x_0) = y_0 \end{cases} \tag{3.36}$$

We can write the equation in the form

$$y'(x) = f(x)g(y(x)$$

as soon as we define $f(x) = 1$ e $g(y) = e^{-y^4} - 1$;

We have $f \in \mathcal{C}^0(\mathbb{R})$ and $g \in \mathcal{C}^1(\mathbb{R})$, and there is one and only one solution for every $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}$. There are constant solutions for the equation that can be found substituting $y(x) = c$ and solving

$$0 = e^{-c^4} - 1$$

We find that there is only one constant solution $y(x) = c = 0$.

If $y_0 = 0$, the constant solution is the unique solution of the Cauchy problem.

Let us suppose now $x_0 = 0$ and $y_0 = 1$. We can assume that $y(x) \neq 0$ near 0, we can separate the variables and integrate on $[0, x]$; we obtain

$$\frac{y'(x)}{e^{-(y(x))^4} - 1} = 1 \tag{3.37}$$

$$\int_0^x \frac{y'(t)}{e^{-(y(t))^4} - 1} dt = \int_0^x dt \tag{3.38}$$

that is

$$\int_1^{y(x)} \frac{ds}{\mathrm{e}^{-s^4} - 1} = x$$

If we define

$$h(y) = \int_1^y \frac{ds}{\mathrm{e}^{-s^4} - 1}$$

we can write the last equation in the form

$$h(y(x)) = x$$

Let us study h.

The integrand is continuous for $s \neq 0$ and

$$\lim_{s \to 0} \frac{1}{e^{-s^4} - 1} = -\infty$$

so it is infinite of order 4, and the improper integral is divergent in 0; it follows that it must be $y > 0$.

Moreover

$$\lim_{s \to +\infty} \frac{1}{e^{-s^4} - 1} = -1$$

so the improper integral is also divergent for $y \to +\infty$.

Finally $h(1) = 0$ e $h'(y) =]frac1e^{-y^4} - 1 < 0$

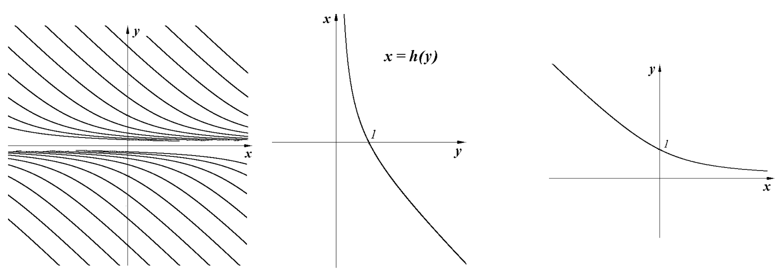
We can also observe that

$$h''(y) = \frac{4y^3 e^{-y^4}}{(e^{-y^4} - 1)^2} > 0$$

for every $y > 0$, so that H is a convex function; moreover

$$\lim_{y \to +\infty} h'(y) = -1$$

and the tangent line to the graph of h approaches the line that bisects the second and fourth quadrants



(a) Grafico 1 (b) Grafico2 (c) Grafico3

Figure 3.6:

Since

$$h(y(x)) = x$$

we have

$$y(x) = h^{-1}(x)$$

as in Figure 3.6

Since the given equation is autonomous if $y(x)$ is a solution, then $y(x + a)$ is also a solution for every $a \in \mathbb{R}$. Moreover we can find solutions when $y < 0$, in similar way.

4. Ordinary Differential Equations

Let $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ be an open set and let $I = [x_0 - a, x_0 + a] \subset \mathbb{R}$ and $A = \{y \in \mathbb{R}^n \ : \ |y - y_0| \leq b\}$ such that $I \times A \subset \Omega$.

Let $f : I \times A \longrightarrow A$ be a continuous function. and let us consider the problem to find $\delta \in \mathbb{R}_+$ and a differentiable function

$$y : I_\delta = (x_0 - \delta, x_0 + \delta) \longrightarrow A$$

such that

$$\begin{cases} y'(x) = f(x, y(x)) & , x \in I_\delta \\ y(x_0) = y_0 \end{cases} \tag{4.1}$$

We say that 4.1 is a Cauchy's problem and that y is a local solution of the Cauchy's problem.

4.1 *Picard's existence and uniqueness theorem for the solution of a Cauchy's problem*

The following theorem proves the existence of a solution for a Cauchy's problem, using the so-called Picard's approximations.

To prove the theorem we need to assume that f satisfies the Lipschitz's condition with respect to the variables in A and uniqueness can be proved too using the Gronwall's Lemma

Theorem 4.1 *Let $f : \Omega \longrightarrow \mathbb{R}^n$, $I \times A \subset \Omega$ $I = [x_0 - a, x_0 + a]$, $A = \{y \in \mathbb{R}^n : \ |y - y_0| \leq b\}$ and let us suppose that:*

- *f is continuous in Ω;*
- $|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \ \forall x \in I, \forall y_1, y_2 \in A$

  Let

$$M = \max\{|f(x, y)| \ : \ (x, y) \in \Omega\} \quad \text{and} \quad \delta = \min\{a, \frac{b}{M}\}$$

.

Then there exists one and only one function

$$y : I_\delta \longrightarrow A$$

solution of the Cauchy's problem.

Proof. First of all we remark that to find $\delta \in \mathbb{R}_+$ and a differentiable function

$$y : I_\delta = (x_0 - \delta, x_0 + \delta) \longrightarrow A$$

such that

$$\begin{cases} y'(x) = f(x, y(x)) \quad , x \in I_\delta \\ y(x_0) = y_0 \end{cases} \tag{4.2}$$

is equivalent to finding $\delta \in \mathbb{R}_+$ and a continuous function

$$y : I_\delta = (x_0 - \delta, x_0 + \delta) \longrightarrow A$$

such that

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \qquad , \qquad x \in I_\delta \tag{4.3}$$

We can obtain 4.3 from 4.2 simply integrating on $[x_0, x]$ both sides of the first equality and keeping in mind that $y(x_0) = y_0$, while we can differentiate both sides of 4.3 and substitute x_0 for x again in 4.3 to obtain 4.2

Let us now define a sequence of functions approximating a solution of Cauchy's problem

$$y_k : I_\delta \longrightarrow A$$

as

$$\begin{cases} y_0(x) = y_0 \\ y_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_k(t))dt \end{cases}$$

y_k is known as $k - th$ Picard's approximation; we prove that y_k is a well-defined sequence of functions and converges to a solution for the Cauchy's problem 4.1, performing several steps

Step 1

y_k is well-defined because we can prove by induction that

$$y_k(x) \in A, \qquad \forall x \in I_\delta \quad , \quad \forall k \in \mathbb{N}$$

Indeed

$$y_0(x) = y_0 \in A \qquad \forall x \in I_\delta$$

moreover if

$$y_k(x) \in A \qquad \forall x \in I_\delta$$

we have

$$|y_{k+1}(x) - y_0| \leq \left| \int_{x_0}^{x} |f(t, y_k(t))|dt \right| \leq M|x - x_0| \leq M\delta \leq b$$

Hence

$$y_{k+1}(x) \in A \qquad \forall x \in I_\delta$$

Step 2

y_k is a uniformly convergent sequence on I_δ.

We have

$$|y_{k+1}(x) - y_k(x)| \le ML^k \frac{|x-x_0|^{k+1}}{(k+1)!}$$

Indeed

$$|y_1(x) - y_0(x)| \le \left| \int_{x_0}^{x} |f(t, y_0)| dt \right| \le M|x - x_0|$$

moreover, if we supppose that

$$|y_k(x) - y_{k-1}(x)| \le ML^{k-1} \frac{|x-x_0|^k}{k!}$$

we obtain

$$\begin{aligned} |y_{k+1}(x) - y_k(x)| &\le \left| \int_{x_0}^{x} |f(t, y_k(t)) - f(t, y_{k-1}(t))| dt \right| \le \\ &\le L \left| \int_{x_0}^{x} |y_k(t) - y_{k-1}(t)| dt \right| \le \\ &\le \frac{ML^k}{k!} \left| \int_{x_0}^{x} |t - x_0|^k dt \right| \le ML^k \frac{|x-x_0|^{k+1}}{(k+1)!} \end{aligned}$$

So we can assert that

$$\begin{aligned} |y_{k+p}(x) - y_k(x)| &\le \sum_{h=1}^{p} |y_{k+h}(x) - y_{k+h-1}(x)| \le \\ &\le \sum_{h=1}^{p} ML^{k+h-1} \frac{|x-x_0|^{k+h}}{(k+h)!} = \\ &= \frac{M}{L} \sum_{h=1}^{p} L^{k+h} \frac{|x-x_0|^{k+h}}{(k+h)!} \le \\ &\le \frac{M}{L} \sum_{i=k+1}^{k+p} \frac{(L\delta)^i}{i!} = E_{k,p} \end{aligned} \quad (4.4)$$

Step 3

y_k is uniformly convergent to some function y on I_δ.

Indeed, since

$$e^{L\delta} = \sum_{i=1}^{+\infty} \frac{(L\delta)^i}{i!}$$

we have $|E_{k,p}| < \varepsilon$ when k is large enough and for every $p \in \mathbb{N}$,

Moreover y is a continuous function on I_δ because it is the uniform limit of a sequence of continuous functions y_k.

Step 4

y is a solution of the Cauchy's problem.

Indeed we can take limit for $p \to +\infty$ in 4.4 and we obtain

$$|y(x) - y_k(x)| \le \frac{M}{L} \sum_{i=k+1}^{+\infty} \frac{(L\delta)^i}{i!} = \frac{M}{L} \left(e^{L\delta} - \sum_{i=0}^{k} \frac{(L\delta)^i}{i!} \right) =$$
$$= \frac{M}{L} e^{\xi} \frac{(L\delta)^{k+1}}{(k+1)!} \le \frac{M}{L} e^{L\delta} \frac{(L\delta)^{k+1}}{(k+1)!} = E_k$$

where $|\xi| \le L\delta$.

Obviously $\lim_k E_k = 0$ and, since

$$y_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_k(t))dt$$

keeping in mind that

$$\left| \int_{x_0}^{x} [f(t, y_k(t)) - f(t, y(t))]dt \right| \le \left| \int_{x_0}^{x} L|y_k(t) - y(t)|dt \right| \le L\delta E_k$$

we obtain, taking limit for $k \to +\infty$, that

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt$$

whence y is a solution of 4.3 and of 4.2 as well..

Step 5

The solution of 4.2 is unique.

If y and z are solutions of the Cauchy's problem by **??** we have

$$|y(x) - z(x)| \le L \left| \int_{x_0}^{x} |y(t) - z(t)|dt \right|$$

and we can conclude using Gronwall's Lemma. □

Remark. In Step 4 we proved that

$$|y_k(x) - y(x)| \le \frac{M}{L} e^{L\delta} \frac{(L\delta)^{k+1}}{(k+1)!}.$$

This inequality is a simple estimate of the error that we commit if we substitute y by y_k. □

4.2 *Peano's existence theorem for a solution of a Cauchy's problem*

Definition 4.1 *Let $f : \Omega \longrightarrow \mathbb{R}^n$, $I \times A \subset \Omega$ $I = [x_0 - a, x_0 + a]$, $A = \{y \in \mathbb{R}^n : |y - y_0| \le b\}$ and let us suppose that f is continuous in Ω;.*

Let

$$M = \max\{|f(x,y)| \ : \ (x,y) \in \Omega\} \quad \textit{and} \quad \delta = \min\{a, \frac{b}{M}\}$$

Let us consider a sequence of functions approximating a solution of the Cauchy problem; we call these functions **Euler's approximations**

Let

$$P_k = \{x_0 < x_1 < x_2 < \cdots < x_{n_k} = x_0 + \delta\}$$

be a partition of the interval $[x_0, x_0 + \delta]$ *and let*

$$\delta_k = \max\{x_{i+1} - x_i\}$$

the maximum step size. Let us also assume that $\delta_k \to 0$

Let

$$y_k : [x_0, x_0 + \delta] \longrightarrow A$$

defined by

$$\begin{cases} y_k(x_0) = y_0 \\ y_k(x) = y_k(x_i) + f(x_i, y_k(x_i))(x - x_i), \quad x_i \leq x \leq x_{i+1} \end{cases}$$

We have $y_k(x_0) = y_0 \in A$; moreover if

$$y_k(x_j) \in A$$

we have

$$|y_k(x_{i+1}) - y_0| \leq M|x_{i+1} - x_0| \leq M\delta \leq b.$$

Therefore y_k is well-defined since

$$y_k(x) \in A \quad \text{if} x \in [x_0, x_0 + \delta]$$

and $f(x_i, y_k(x_i))$ can be calculated.

Since y_k is a polygonal curve and the slope of its line segments does not exceed M it is

$$|y_k(\xi) - y_k(\eta)| \leq M(\xi - \eta) \qquad \forall \xi, \eta \in [x_0, x_0 + \delta]$$

Using Euler approximations Peano proved the existence of a solution for a Cauchy's problem. More precisely he proved that the sequence of Ruler's approximations converges to a differentiable function $y : [x_0 - \delta, x_0 + \delta] \longrightarrow A$, such that

$$\begin{cases} y'(x) = f(x, y(x)), \quad x \in [x_0 - \delta, x_0 + \delta] \\ y(x_0) = y_0 \end{cases}$$

In order to prove Peano's theorem ne need some preliminaries.

Definition 4.2 *Let $f_k : [a,b] \longrightarrow \mathbb{R}^n$ be a sequence of functions over a closed and bounded interval; f_k is uniformly bounded if there exists $M \in \mathbb{R}$ such that*

$$\|f_k(x)\| \leq M \ \ \forall x \in [a,b] \ , \ \forall k \in \mathbb{N};$$

f_k is equicontinuous if $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ such that when $|x-y| < \delta_\varepsilon$, $x, y \in [a,b]$ then $\|f_k(x) - f_k(y)\| < \varepsilon \ \forall k \in \mathbb{N}$.

We can prove the following compactness theorem for uniformly bounded, equicontinuous sequences of functions.

Remark. The sequence of **Euler's approximations** is uniformly bounded and equicontinuos. □

Theorem 4.2 *- Ascoli-Arzelà - Let $f_k : [a,b] \longrightarrow \mathbb{R}^n$ be a uniformly bounded, equicontinuous sequence of functions over a closed, bounded interval $[a,b]$*

The there exists a subequence $f_{k'}$ of f_k and exists a continuous function $f : [a,b] \longrightarrow \mathbb{R}^n$ such that $f_{k'}$ is uniformly convergent to f over $[a,b]$.

Proof. Sia

$$D = \{x_i \ : \ i \in N\}$$

be a countable subset dense in $[a,b]$, (e.g. $D = \{x \in \mathbb{Q} \ : \ a \leq x \leq b\}$).

Since f_k is uniformly bounded we can choose a subsequence $f_{h_1} = f_{k_{h_1}}$ of f_k such that

$$f_{h_1}(x_1) \to \alpha_1$$

then we can find a subsequence $f_{h_2} = f_{h_{1_{h_2}}}$ of f_{h_1} such that

$$f_{h_2}(x_2) \to \alpha_2$$

It also results

$$f_{h_2}(x_1) \to \alpha_1$$

By recurrence, there is a subsequence $f_{h_j} = f_{h_{j-1_{h_j}}}$ of $f_{h_{j-1}}$ such that

$$f_{h_j}(x_j) \to \alpha_j$$

and we have

$$f_{h_j}(x_i) \to \alpha_i \quad per \ \ i = 1, 2, ..., j \ .$$

Now let $f_{k'}$ the $k-$th term of the $k-$th subsequence f_{h_k} (i.e. the term obtained substituting k for h_k in f_{h_k}).

We have

$$f_{k'}(x_j) = \!\!\longrightarrow \alpha_j \quad \forall j \in \mathbb{N}. \tag{4.5}$$

as soon as we recall that k' is eventually greater than j, $\forall j \in \mathbb{N}$)

Let us now prove that $f_{k'}$ satisfies the Cauchy's Criterion for uniform convergence in $[a,b]$.

Let $\varepsilon > 0$ and let $y_i \in [a,b]$ such that

$$a = y_0 < y_1 < .... < y_i < y_{i+1} < ...... < y_m = b$$

and

$$|y_i - y_{i-1}| < \delta_{\varepsilon/3}$$

(where δ_ε is chosen in accordance with the definition of equicontinuity).

Since D is dense in $[a, b]$ we can find

$$\xi_i \in [y_{i-1}, y_i] \cap D$$

moreover, since there are finitely many ξ_i, for $k \in \mathbb{N}$ sufficiently large we have

$$\|f_{(k+p)'}(\xi_i) - f_{k'}(\xi_i)\| \le \varepsilon/3 \quad \forall p \in \mathbb{N}\ ,\ i = 1, 2, ..., m$$

For every $x \in [a, b]$, we have $x \in [y_{i-1}, y_i]$ (for suitable i) and therefore, using equicontinuity

$$\begin{aligned} \|f_{(k+p)'}(x) - f_{k'}(x)\| \le \|f_{(k+p)'}(x) - f_{(k+p)'}(\xi_i)\| + \\ + \|f_{(k+p)'}(\xi_i) - f_{k'}(\xi_i)\| + \|f_{k'}(\xi_i) - f_{k'}(x)\| \le \\ \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \end{aligned}$$

At last $f_{k'}$ satisfies uniform Cauchy's criterion and us uniformly convergent to some function f. □

Let us now prove the Peano's existence theorem.

Theorem 4.3 *- Peano - Let $I = [x_0 - a, x_0 + a]$, $A = \{y \in \mathbb{R}^n \ :\ |y - y_0| \le b\}$, and let $f : I \times A \longrightarrow A$ be a continuous function and let.*

$$M = \max\{|f(x, y)| : (x, y) \in I \times A\} \qquad , \qquad \delta = \min\{a, \frac{b}{M}\}$$

Then there is a differentiable function $y : [x_0 - \delta, x_0 + \delta] \longrightarrow A$, such that

$$\begin{cases} y'(x) = f(x, y(x)), & x \in [x_0 - \delta, x_0 + \delta] \\ y(x_0) = y_0 \end{cases}$$

Proof. Without any loss of generality, we look for a forward solution, i'e'a solution defined in $[x_0, x_0 + \delta]$

Let y_k be the sequence of forward Euler's approximations.

We already observed that y_k is a uniformly bounded equicontinuous sequence, so, by Ascoli-Arzelà's theorem, there is a subsequence of y_k uniformly convergent to some function $y : [x_0, x_0 + \delta] \longrightarrow A$.

As a consequence y is a continuous function and moreover y is a solution of the Cauchy's problem..

Let $\varepsilon_k \to 0$, since f is uniformly continuous on $I \times A$, There exists γ_k such that, if

$$|x' - x''| < \gamma_k \qquad , \qquad \|y' - y''\| < \gamma_k$$

then

$$\|f(x', y') - f(x'', y'')\| < \varepsilon_k.$$

Now, let us choose the maximum step size δ_k such that

$$\delta_k \leq \min\{\gamma_k, \gamma_k / M\}$$

When $x \in [x_i, x_{i+1}]$ then $|x - x_i| < \delta_k$ and, since the maximum slope of Euler's approxximation is M, we can assert that

$$\|y_k(x) - y_k(x_i)\| \leq M\delta_k$$

moreover, by equicontinuity,

$$\|f(x, y_k(x)) - f(x_i, y_k(x_i))\| \leq \varepsilon_k.$$

Therefore for $x \in [x_i, x_{i+1}]$ si ha

$$y_k'(x) = f(x_i, y_k(x_i)) - f(x, y_k(x)) + f(x, y_k(x)) = \\ = f(x, y_k(x)) + \Delta_k(x)$$

where

$$\|\Delta_k(x)\| = \|f(x, y_k(x)) - f(x_i, y_k(x_i))\| < \varepsilon_k$$

and the estimate in independent from i.

So we can assert that

$$y_k(x) = y_0 + \int_{x_0}^{x} y_k'(t)dt \;= \\ = y_0 + \int_{x_0}^{x} f(t, y_k(t))dt \; + \int_{x_0}^{x} \Delta_k(t)dt$$

Since y_k is uniformly convergent to y we have

$$\int_{x_0}^{x} f(t, y_k(t)dt \;\longrightarrow\; \int_{x_0}^{x} f(t, y(t))dt$$

uniformly in $[x_0, x_0 + \delta]$ and we have

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \;.$$

□

Remark. We proved that the sequence of Euler's approximations has a subsequence which uniformly converges to a solution of the Cauchy's problem.

We can also prove, with the same arguments, that every subsequence of Euler's approximations has a subsequence which uniformly converges to a solution of the Cauchy's problem.

Moreover, when uniqueness of the solution in guaranteed, we can assert that the limit is always the unique solution of the Cauchy's problem.

Therefore a well known results on sequences allow us to state that the whole sequence of Euler approximations uniformly converges to the solution of the Cauchy's problem.

□

Under suitable assumptions we can estimate the error in Euler's approximations .

Theorem 4.4 *Let a_n be a sequence such that there exists $\ell \in \mathbb{R}^*$ such that for every subsequence b_k extracted from a_n there is a subsequence c_h extracted from b_k such that*

$$\lim_h c_h = \ell$$

Then

$$\lim_n a_n = \ell$$

Proof. If it were

$$\lim_n a_n \neq \ell$$

we colud find $\varepsilon_0 > 0$ and b_k extracted from a_n such that $b_k \notin I(\ell, \varepsilon_0)$

by assumptions, we can find a subsequence c_h such that

$$\lim_h c_h = \ell$$

and this is a contradiction because we should have

$$c_h \in I(\ell, \varepsilon_0) \qquad \text{and} \qquad c_h = b_{k_h} \notin I(\ell, \varepsilon_0)$$

for h sufficiently large. □

Theorem 4.5 *Let $I = [x_0 - a, x_0 + a]$, $A = \{y \in \mathbb{R}^n \ : \ |y - y_0| \leq b\}$, and let $f : I \times A \longrightarrow A$ be a continuous function and let.*

$$M = \max\{|f(x,y)| : (x,y) \in I \times A\} \qquad , \qquad \delta = \min\{a, \frac{b}{M}\}$$

Let moreover assume that there are L_x ed L_y such that:

$$\left|\frac{\partial f}{\partial x}(x,y)\right| \leq L_x \quad , \quad |\nabla_y f(x,y)| \leq L_y$$

Then

$$|y_k(x) - y(x)| \leq \delta_k (L_x + L_y M)(x - x_0) e^{L_y(x - x_0)}/2$$

Proof. Si ha

$$|y_k(x) - y_h(x)| \leq \left|\int_{x_0}^x [f(t, y_k(t)) - f(t, y_h(t))]dt\right| + \\ + \left|\int_{x_0}^x \Delta_k(t)dt\right| + \left|\int_{x_0}^x \Delta_h(t)dt\right|$$

where

$$\Delta_k(t) = |f(t, y_k(t)) - f(t, y_k(x_i))|$$

Since, by the assumptions,

$$|\Delta_k(x)| \leq L_x|x - x_i| + L_y|y_k(x) - y_k(x_i)| \leq \\ \leq L_x|x - x_i| + L_y M|x - x_i| = \\ = (L_x + L_y M)|x - x_i|$$

we obtain

$$|y_k(x) - y_h(x)| \leq (L_x + L_y M)(\delta_k + \delta_h)(x - x_0)/2 + \\ + \left|\int_{x_0}^x L_y|y_k(t) - y_h(t)|dt\right|$$

We can finally use the Gronwall's Lemma to obtain

$$|y_k(x) - y_h(x)| \leq (\delta_k + \delta_h)(L_x + L_y M)\frac{x - x_0}{2}e^{L_y(x-x_0)}.$$

Therefore y_k is uniformly convergent do some function y and we can use the same arguments as in Peano's theorem to assert that it is a solution of the Cauchy's problem. Letting $h \to +\infty$ we prove the theorem. □ Cauchy.

4.3 *Numerical methods for the solution of a Cauchy's problem.*

Euler's method is a one-step, first-order integration method because it is based upon fist order Taylor expansion of the solution and it uses at any time level only information from the previous one.

We can also obtain one-step, higher-order methods using higher order Taylor expansions; we can derive a second-order method as follows.

Let us consider the Cauchy's problem

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$$

and let us expand y using second-order Taylor Formula

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2}y''(x) + h^2\omega(h)$$

Since

$$y''(x) = f_x(x, y(x)) + f_y(x, y(x))y'(x) = f_x(x, y(x)) + f_y(x, y(x))f(x, y(x))$$

we get

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2}y''(x) + h^2\omega(h)$$

and

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2}(f_x(x, y(x)) + f_y(x, y(x))f(x, y(x))) + h^2\omega(h) \tag{4.6}$$

Now, let us look for α, β, w_1, w_2 such that

$$\begin{cases} k_1 = hf(x_n, y_n) \\ k_2 = hf(x_n + \alpha h, y_n + \beta k_1) \\ y_{n+1} = y_n + w_1 k_1 + w_2 k_2 \end{cases}$$

We have

$$y_{n+1} = y_n + w_1 hf(x_n, y_n) + w_2 hf(x_n + \alpha h, y_n + \beta k_1)$$

Substituting $f(x_n + \alpha h, y_n + \beta k_1)$ with its Taylor expansion

$$y_{n+1} = y_n + w_1 h f(x_n, y_n) + w_2 h(f(x_n, y_n) + \alpha h f_x(x_n, y_n) + \beta k_1 f_y(x_n, y_n) + h\omega(h))$$

whence

$$y_{n+1} = y_n + (w_1 + w2) h f(x_n, y_n) + w_2 h(\alpha h f_x(x_n, y_n) + \beta k_1 f_y(x_n, y_n) + h\omega(h))$$

and

$$y_{n+1} = y_n + (w_1 + w2) h f(x_n, y_n) + w_2 h^2(\alpha f_x(x_n, y_n) + \beta f(x_n, y_n) f_y(x_n, y_n)) + h^2\omega(h) \tag{4.7}$$

Comparing 4.6 and 4.7 we deduce that it must be

$$\begin{cases} w_1 + w_2 = 1 \\ w_2\alpha = \frac{1}{2} \\ w_2\beta = \frac{1}{2} \end{cases} \tag{4.8}$$

The system is underdetermined and it ha infinitely many solutions.

One possible solution is

$$w_1 = w_2 = \frac{1}{2} \quad , \quad \alpha = \beta = 1$$

and, in this case, we obtain the modified Euler's method:

$$\begin{aligned} x_{n+1} &= x_n + h \\ k_1 &= h f(x_n, y_n) \\ k_2 &= h f(x_n + h, y_n + k_1) \\ y_{n+1} &= y_n + (k_1 + k_2)/2 \end{aligned}$$

If we choose a different set of solutions of system 4.8 we obtain a different second order one-step integration method.

Moreover, if we use higher order Taylor expansions, we can find higher order integration method. The most popular among them is the Runge-Kutta fourth-order method which is defined as follows.

$$\begin{aligned} x_{n+1} &= x_n + h \\ k_1 &= h f(x_n, y_n) \\ k_2 &= h f(x_n + h/2, y_n + k_1/2) \\ k_3 &= h f(x_n + h/2, y_n + k_2/2) \\ k_4 &= h f(x_n + h, y_n + k_3) \\ y_{n+1} &= y_n + (k_1 + 2k_2 + 2k_3 + k_4)/6 \end{aligned}$$

Some popular one-step integration methods are listed below.

- Euler (order 1)

$$\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + hf(x_n, y_n) \end{aligned}$$

- Modified Euler (order 2)

$$\begin{aligned} x_{n+1} &= x_n + h \\ k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + h, y_n + k_1) \\ y_{n+1} &= y_n + (k_1 + k_2)/2 \end{aligned}$$

- Heun (order 3)

$$\begin{aligned} x_{n+1} &= x_n + h \\ k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + h/3, y_n + k_1/3) \\ k_3 &= hf(x_n + 2h/3, y_n + 2k_2/3) \\ y_{n+1} &= y_n + (k_1 + 3k_3)/4 \end{aligned}$$

- Kutta (order 3)

$$\begin{aligned} x_{n+1} &= x_n + h \\ k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + h/2, y_n + k_1/2) \\ k_3 &= hf(x_n + h, y_n + 2k_2 - k_1) \\ y_{n+1} &= y_n + (k_1 + 4k_2 + k_3)/6 \end{aligned}$$

- Runge-Kutta (order 4)

$$\begin{aligned} x_{n+1} &= x_n + h \\ k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + h/2, y_n + k_1/2) \\ k_3 &= hf(x_n + h/2, y_n + k_2/2) \\ k_4 &= hf(x_n + h, y_n + k_3) \\ y_{n+1} &= y_n + (k_1 + 2k_2 + 2k_3 + k_4)/6 \end{aligned}$$

- Runge-Kutta (order 4)

$$\begin{aligned} x_{n+1} &= x_n + h \\ k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + h/3, y_n + k_1/3) \\ k_3 &= hf(x_n + 2h/3, y_n + k_2 - k_1/3) \\ k_4 &= hf(x_n + h, y_n + k_1 - k_2 + k_3) \\ y_{n+1} &= y_n + (k_1 + 3k_2 + 3k_3 + k_4)/8 \end{aligned}$$

4.4 *Error estimate for the Euler's method*

Let $I = [x_0 - a, x_0 + a]$, $A = \{y \in \mathbb{R}^n \ : \ |y - y_0| \leq b\}$, $\Omega = I \times A$ and let $f : \Omega \longrightarrow A$, $f \in \mathcal{C}^2(\Omega)$.

$$M = \max\{|f(x,y)| : (x,y) \in I \times A\} \qquad , \qquad \delta = \min\{a, \frac{b}{M}\}$$

Let moreover assume that there are L_x ed L_y such that:

$$\left|\frac{\partial f}{\partial x}(x,y)\right| \leq L_x \quad , \quad |\nabla_y f(x,y)| \leq L_y$$

Let us consider the Cauchy's problem:

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$$

Let $h > 0$ and let

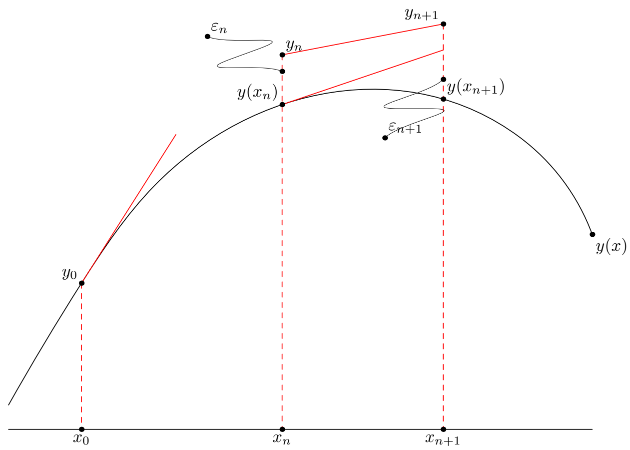
$$\begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + hf(x_n, y_n) \end{cases}$$

be the vertices of the Euler's polygonal line.

Then if y is the solution of the Cauchy's problem we have

$$y''(x) = f_x(x, y(x)) + f_y(x, y(x))y'(x) = \\ = f_x(x, y(x)) + f_y(x, y(x))f(x, y(x))$$

and we can assert that $y \in \mathcal{C}^2((x_0 - \delta, x_0 + \delta)$,



We can expand y using the Taylor's formula

$$y(x_{n+1}) = y(x_n) + hf(x_n, y(x_n)) + \frac{1}{2}h^2 y''(\xi)$$

with $\xi \in (x_n, x_{n+1})$. and we can estimate

$$E = \frac{1}{2}h^2 \|y''(\xi)\| \leq \frac{1}{2}h^2(L_x + ML_y)$$

Let us define

$$\varepsilon_n = \|y(x_n) - y_n\| \qquad , \qquad \varepsilon_0 = 0$$

We have

$$y(x_{n+1}) - y_{n+1} = y(x_n) - y_n + hf(x_n, y(x_n)) - hf(x_n, y_n) + \frac{1}{2}h^2 y''(\xi)$$

whence

$$\varepsilon_{n+1} = \|y(x_{n+1}) - y_{n+1}\| = \|y(x_n) - y_n\| + h\|f(x_n, y(x_n)) - f(x_n, y_n)\| + E \leq \\ \leq \|y(x_n) - y_n\| + h\|f_y(x_n, \eta_n)\| \|y(x_n) - y_n\| + E \\ = \varepsilon_n + hL_y\varepsilon_n + E$$

Therefore

$$\begin{cases} \varepsilon_{n+1} \leq (1 + hL_y)\varepsilon_n + E \\ \varepsilon_0 = 0 \end{cases}$$

If we perform some iteration steps we get

$$\begin{aligned}\varepsilon_1 &\le E\\ \varepsilon_2 &\le (1+hL_y)\varepsilon_1 + E \le E((1+hL_y)+1)\\ \varepsilon_3 &\le (1+hL_y)\varepsilon_2 + E \le E((1+hL_y)^2 + (1+hL_y)+1)\end{aligned}$$

and we can deduce that

$$\begin{aligned}\varepsilon_n \le E(1 + (1+hL_y) + (1+hL_y)^2 + \cdots + (1+hL_y)^{n-1})\\ = E\frac{1-(1+hL_y)^n}{1-(1+hL_y)} = E\frac{(1+hL_y)^n - 1}{hL_y} \le\\ \le \frac{1}{2}h(L_x + ML_y)\frac{(1+hL_y)^n - 1}{L_y} \quad (4.9)\end{aligned}$$

From $e^x \ge 1 + x$ we get

$$e^{nhL_y} = \left(e^{hl_y}\right)^n \ge (1+hL_y)^n$$

and

$$\varepsilon_n \le \frac{1}{2}(L_x + ML_y)\frac{e^{nhL_y} - 1}{L_y} = \frac{1}{2}h(L_x + ML_y)\frac{e^{(x-x_0)L_y} - 1}{L_y}$$

as soon as we recall that $x_n = x_0 + nh$.

So far we have neglected the truncation error, i.e. the error due to the use of a decimal approximation in place of the number itself. Let τ be the error introduced by such an approximation, we have

$$\begin{cases}\varepsilon_{n+1} \le (1+hL_y)\varepsilon_n + (E+\tau)\\ \varepsilon_0 = 0\end{cases}$$

If we proceed as we have done before, we get

$$\varepsilon_n \le (E+\tau)\frac{e^{(x-x_0)L_y} - 1}{L_y}\left(\frac{1}{2}h(L_x + ML_y) + \frac{\tau}{h}\right)\frac{e^{(x-x_0)L_y} - 1}{L_y}$$

We can choose

$$h = \sqrt{\frac{2\tau}{L_x + ML_y}}$$

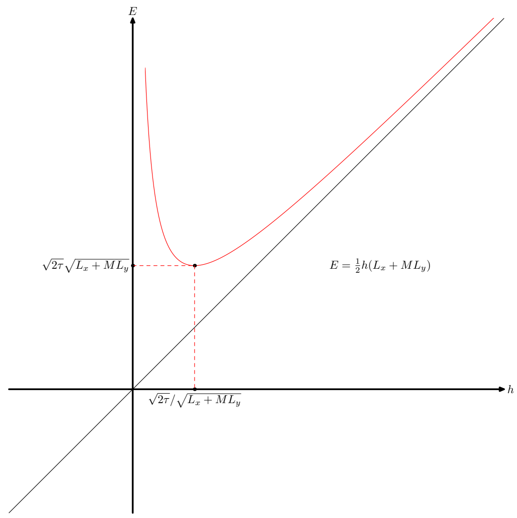
so that the term

$$\left(\frac{1}{2}h(L_x + ML_y) + \frac{\tau}{h}\right)$$

reaches its minimum value

$$\sqrt{2\tau}\sqrt{L_x + ML_y}$$

This remark allows us to make the optimum choice of the integration stepsize h

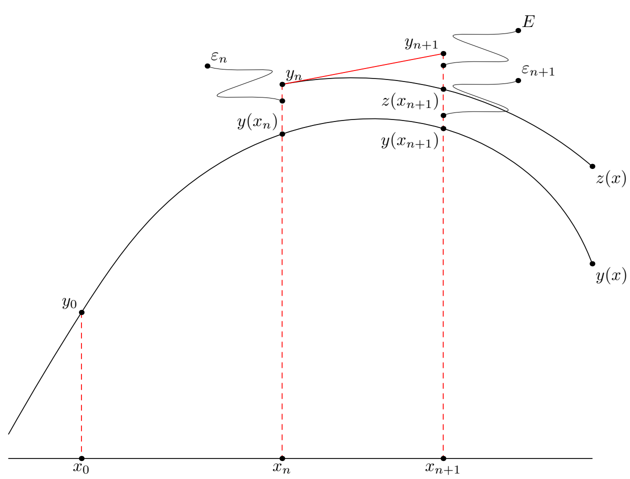


4.5 *Error estimate for generic one-step integration methods*

Let (x_n, y_n) the vertices of an approximating polygonal line for the solution of the Cauchy's problem and let z the exact solution of the Cauchy's problem with initial data $z(x_n) = y_n$.

Let us moreover suppose that we are able to estimate

$$\|z(x_{n+1}) - y_{n+1}\| \le E$$



We define

$$\varepsilon_n = \|y(x_n) - y_n\|$$

and we get

$$\begin{aligned}\|y(x_{n+1}) - y_{n+1}\| = \|y(x_{n+1}) - z(x_{n+1}) + z(x_{n+1}) - y_{n+1}\| \le \\ \le \|y(x_{n+1}) - z(x_{n+1})\| + \|z(x_{n+1}) - y_{n+1}\| \le \\ \le \|y(x_{n+1}) - z(x_{n+1})\| + E\end{aligned}$$

By Gronwall's lemma,

$$\|y(x_{n+1}) - z(x_{n+1})\| \leq \|y(x_n) - z(x_n)\| e^{hL_y} \tag{4.10}$$

and

$$\begin{cases} \varepsilon_{n+1} \leq e^{hL_y} \varepsilon_n + E \\ \varepsilon_0 = 0 \end{cases}$$

whence

$$\begin{aligned} \varepsilon_n \leq E(1 + e^{hL_y} + (e^{hL_y})^2 + \cdots + (e^{hL_y})^{n-1} = \\ = E\frac{(e^{hL_y})^n - 1}{e^{hL_y} - 1} = E\frac{e^{nhL_y} - 1}{e^{hL_y} - 1} = E\frac{e^{L_y(x-x_0)} - 1}{e^{hL_y} - 1} \end{aligned}$$

For Euler's method we can estimate E as follows

$$E = \leq \frac{1}{2}h^2(L_x + ML_y)$$

and

$$\begin{aligned} \varepsilon_n \leq \frac{1}{2}h^2(L_x + ML_y)\frac{e^{L_y(x-x_0)} - 1}{e^{hL_y} - 1} = \\ = \frac{1}{2}h(L_x + ML_y)\frac{h}{e^{hL_y} - 1}(e^{L_y(x-x_0)} - 1) \end{aligned} \tag{4.11}$$

4.6 Prolongability of solutions of Cauchy's problems.

Let $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ be an open set and let $I = [x_0 - a, x_0 + a] \subset \mathbb{R}$ and $A = \{y \in \mathbb{R}^n \ : \ |y - y_0| \leq b\}$ such that $I \times A \subset \Omega$.

Let $f : I \times A \longrightarrow A$ be a continuous function. and let us consider the Cauchy's problem

$$\begin{cases} y'(x) = f(x, y(x)) \quad , x \in I_\delta \\ y(x_0) = y_0 \end{cases} \tag{4.12}$$

Definition 4.3 *Let $y : [a, b) \longrightarrow A$ a solution of the Cauchy's problem 4.12. Then y right prolongable if exists*

$$z : [a, c) \longrightarrow A$$

such that $c > b$ and $y(x) = z(x)$ $\forall x \in [a, b)$.

By Gronwall's lemma we can prove the following theorem:

Theorem 4.6 *Let us consider the Cauchy's problem 4.12 and let us suppose that*

1. f is continuous in Ω ;

2. $\|f(x,y_1) - f(x,y_2)\| \leq L\|y_1 - y_2\| \; \forall x \in I, \forall y_1, y_2 \in A$.

Let y , z solutions of the Cauchy's problem in I_1 ed I_2 respectively. then

$$y(x) = z(x) \quad \forall x \in I_1 \cap I_2.$$

Proof. For every $\varepsilon > 0$ we have

$$\|y(x) - z(x)\| \leq \int_{x_0}^{x} L\|y(t) - z(t)\|dt + \varepsilon \qquad \forall x \in I_1 \cap I_2$$

We can apply Gronwall's lemma and we conclude letting $\varepsilon \to 0$. □

We can directly prove an existence theorem for a global solution of the Cauchy's problem 4.12

Theorem 4.7 *Let $I = (x_0 - a, x_0 + a)$, $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and let us assume that f is a continuous locally Lipschitz function, i.e. there exists $L > 0$ such that for every bounded set $B \subset I \times \mathbb{R}^n$*

$$\|f(x,y_1) - f(x,y_2)\| \leq L\|y_1 - y_2\|$$

$\forall x \in I$, $\forall y_1, y_2 \in \mathbb{R}^n$.

Let us moreover assume that one of the following conditions holds:

1. *f is bounded on the strip $I \times \mathbb{R}^n$, i.e.*

$$\|f(x,y)\| \leq M \;\; \forall x \in I \; , \; \forall y \in \mathbb{R}^n$$

2. *f is globally Lipschitz on the strip $I \times \mathbb{R}^n$, i.e.*

$$\|f(x,y_1) - f(x,y_2)\| \leq L\|y_1 - y_2\| \; , \; \forall x \in I \; , \; \forall y_1, y_2 \in \mathbb{R}^n$$

Then there is one and only one solution of the Cauchy's problem 4.12 which is defined on all of I.

Proof. When f is bounded on the strip $I \times \mathbb{R}^n$, we can apply Theorem 4.1 choosing $b > Ma$ so that it results

$$\delta = \min\{a, \frac{b}{M}\} = a$$

On the other side, when f is globally Lipschitz on the strip $I \times \mathbb{R}^n$, we can define

$$H = \max\{|f(x,y_0)| \; : \; x \in I\}$$

and we can choose $b > H$.

We have

$$\|f(x,y)\| = \|f(x,y) - f(x,y_0) + f(x,y_0)\| \leq \|f(x,y) - f(x,y_0)\| + \|f(x,y_0)\| \leq L\|y - y_0| + H \leq Lb + H$$

whence

$$\frac{\|f(x,y)\|}{b} = \leq L + \frac{H}{b} \leq L + 1$$

It follows that

$$\frac{b}{M} \geq \frac{1}{L+1}$$

and we can find a solution defined in all of I simply applying theorem 4.1 a finite number of times. □

A more general theorem on prologability of the solution of a Cauchy's problen can also be proved. To this end we have to introduce the distance between two sets in $\mathbb{R}^n$.

Definition 4.4 *Let $A, B \subset \mathbb{R}^n$, we define the distance between A and B as*

$$d(A, B) = inf\{\|x - y\| \ : \ x \in A \ , \ y \in B \ \}.$$

We can prove that

Lemma 4.1 *Let $A \subset \mathbb{R}^n$, a compact set and let $B \subset \mathbb{R}^n$ a closed set such thtat $A \cap B = \emptyset$; then*

$$d(A, B) > 0.$$

Proof. If it were

$$d(A, B) = inf\{\|x - y\| \ : \ x \in A \ , \ y \in B \ \} = 0;$$

we could find $x_n \in A$ e $y_n \in B$ such that $\|x_n - y_n\| \to 0$.

Since A is a compact set we can find $x_{n_k} \to x \in A$ and consequently $y_{n_k} \to x \in B$ which is false since $A \cap B = \emptyset$. □

Theorem 4.8 *Let us consider the Cauchy's problem 4.12 and let us suppose that*

1. *f is continuous in Ω ;*

2. $\|f(x, y_1) - f(x, y_2)\| \leq L\|y_1 - y_2\| \ \forall x \in I \ , \forall y_1, y_2 \in A$.

Let y be a solution of the Cauchy's problem in $[a, b)$ and let

$$\Gamma = \{(x, y(x)) \ : \ x \in [a, b) \ \}$$

Then the following conditions are equivalent:

a) y is right prolongable;

b) Γ is bounded and $d(\Gamma, \partial\Omega) > 0$.

Proof. $a) \Rightarrow b)$. Since y is right prolongable there exists a solution $z : [a, c) \longrightarrow \mathbb{R}$, $c > b$ of the Cauchy's problem, such that

$$y(x) = z(x) \qquad \forall x \in [a, b)$$

and we can define $y(b)$, by continuity.

Let
$$G = \{(x, y(x)) \ : \ x \in [a, b] \ \}$$
then $G = \mathrm{cl}\,\Gamma$ e G and it is a compact set.

Therefore $\mathrm{cl}\,\Gamma$ and Γ is a bounded set and moreover, by the preceding lemma,
$$d(\Gamma, \partial\Omega) > d(G, \partial\Omega) > 0$$
because
$$G \subset \Omega \qquad \text{and} \partial\Omega \subset \mathrm{cl}\,\Omega^c = \Omega^c$$
.

$b) \Rightarrow a)$. Since $0 < d(\Gamma, \partial\Omega) = d(cl\Gamma, \partial\Omega)$ we have
$$\mathrm{cl}\,\Gamma \cap \partial\Omega = \varnothing$$
moreover
$$cl\Gamma \subset cl\Omega = \partial\Omega \cup \Omega.$$
so that
$$cl\Gamma \subset \Omega,$$
If we set
$$M = max\{f(x, y) \ : \ (x, y) \in cl\Gamma\}.$$
we have
$$\|y(x_1) - y(x_2)\| \le M|x_1 - x_2\|$$
and we can easily see that y satisfies the Cauchy's criterion in b^-.

Therefore we can define
$$y(b) = \lim_{x \to b^-} y(x);$$
we also have
$$y'_-(b) = \lim_{x \to b^-} y'(x) = \lim_{x \to b^-} f(x, y(x)) = f(b, y(b))$$
and y is a solution of the Cauchy's problem in $[a, b]$.

Since $(b, y(b)) \in \mathrm{cl}\,\Gamma \subset \Omega$ and Ω is an open set, there is $r > 0$, such that $(b, y(b)) + S(0, r) \subset \Omega$

Using the existence and uniqueness theorem for Cauchy's problem, we can assert that there exists $z : (b - \delta, b + \delta) \longrightarrow A$ such that
$$\begin{cases} z'(x) = f(x, z(x)) \\ z(b) = y(b) \end{cases}$$
and we conclude that
$$\begin{cases} y(x) & , x \in [a, b) \\ z(x) & , x \in [b, b + \delta) \end{cases}$$
is a right prolongation of the solution. □

We can deduce theorem 4.7 as a consequence of theorem 4.8.

Theorem 4.9 *Let* $I = (x_0 - a, x_0 + a)$, $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *and let us assume that* f *is a continuous locally Lipschitz function, i.e. there exists* $L > 0$ *such that for every bounded set* $B \subset I \times \mathbb{R}^n$

$$\|f(x, y_1) - f(x, y_2)\| \le L\|y_1 - y_2\|$$

$\forall x \in I$, $\forall y_1, y_2 \in \mathbb{R}^n$.

Let us moreover assume that one of the following conditions holds:

1. $\|f(x, y)\| \le M \;\; \forall x \in I$, $\forall y \in \mathbb{R}^n$,
2. $\|f(x, y_1) - f(x, y_2)\| \le L\|y_1 - y_2\| \; \forall x \in I$, $\forall y_1, y_2 \in \mathbb{R}^n$.

Then there is one and only one solution of the Cauchy's problem 4.12 which is defined on all of I.

Proof. - 1) Let us assume that the first group of assumptions holds and let $y : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}^n$, $\delta < a$ a solution of the Cauchy's problem 4.12; We have that

$$\Gamma = \{(x, y(x)) \,,\; x \in (x_0 - \delta, x_0 + \delta);$$

is bounded and $d(\Gamma, \partial\Omega) > 0$; Theorem 4.8 allow us to conclude.

-2) Let us assume that the second group of assumptions holds and let $y : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}^n$, $\delta < a$ a solution of the Cauchy's problem 4.12; we have

$$\begin{aligned} \|y(x) - y_0\| &= \left| \int_{x_0}^{x} f(t, y(t))dt \right| \le \\ &\le \left| \int_{x_0}^{x} \|f(t, y(t)) - f(t, y_0)\|dt \right| + \left| \int_{x_0}^{x} \|f(t, y_0)\|dt \right| \le \\ &\le \left| \int_{x_0}^{x} L\|y(t) - y_0\|dt \right| + \left| \int_{x_0}^{x} \|f(t, y_0)\|dt \right| \end{aligned}$$

and, by Gronwall's lemma

$$\|y(x) - y_0\| \le \left| \int_{x_0}^{x} \|f(t, y_0)\|dt \right| e^{L|x - x_0|}.$$

Therefore y is a bounded function and $d(\Gamma, \partial\Omega) > 0$ as long as $\delta < a$.

Theorem 4.8 applies again. □

4.7 Continuous dependance from data

In this section we prove that, under suitable assumptions, small changes in the data cause small changes in the solution.

This topic is clearly crucial because when using mathematical modeling approximated data are to be expected and it is important that data errors do not affect seriously the solution.

Theorem 4.10 *Let* $f : \Omega \longrightarrow \mathbb{R}^n$, $I \times A \subset \Omega$ $I = [x_0 - a, x_0 + a]$, $A = \{y \in \mathbb{R}^n : |y - y_0| \leq b\}$ *and let us suppose that:*

- f *is continuous in* Ω;
- $|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$ $\forall x \in I$, $\forall y_1, y_2 \in A$

  Let

$$M = \max\{|f(x,y)| \ : \ (x,y) \in \Omega\} \quad and \quad \delta = \min\{a, \frac{b}{M}\}$$

.

Let y *be the solution of the Cauchy's problem*

$$\begin{cases} y'(x) = f(x, y(x)) & , x \in I_\delta = [x_0 - \delta, x_0 + \delta] \\ y(x_0) = y_0 \end{cases}$$

and let z *be the solution of the Cauchy's problem*

$$\begin{cases} z'(x) = f(x, z(x)) & , x \in I_{\delta_1} \\ z(x_0) = z_0 \end{cases}$$

then we have
$\|y(x) - z(x)\| \leq \|y_0 - z_0\| e^{L|x - x_0|}$ $\forall x \in I_{\delta_1} \cap I_\delta$.

Moreover let $g : I \times A \longrightarrow \mathbb{R}^n$ *and let* w *be a solution of the Cauchy's problem*

$$\begin{cases} w'(x) = g(x, w(x)) & , x \in I_{\delta_2} \\ w(x_0) = y_0 \end{cases}$$

then we have
$\|y(x) - w(x)\| \leq \left| \int_{x_0}^x \|f(t, w(t)) - g(t, w(t))\| dt \right| e^{L|x - x_0|}$,
$\forall x \in I_{\delta_2} \cap I_\delta$.

Finally let $\xi_0 \in I$ *and let* u *be a solution of the Cauchy's problem*

$$\begin{cases} u'(x) = f(x, u(x)) & , x \in I_{\delta_3} \\ u(\xi_0) = y_0 \end{cases}$$

then we have

$$\|y(x) - u(x)\| \leq M|\xi_0 - x_0| e^{L|x - x_0|} \ , \ \forall x \in I_{\delta_3} \cap I_\delta.$$

Proof. As to the first case, we have

$$\|y(x) - z(x)\| \leq \|y_0 - z_0\| + \left| \int_{x_0}^x \|f(t, y(t)) - f(t, z(t))\| dt \right| \leq$$
$$\leq \|y_0 - z_0\| + \left| \int_{x_0}^x L\|y(t) - z(t)\| dt \right|$$

We conclude applying Gronwall's lemma.

As to the second case,we have

$$\begin{aligned}\|y(x)-w(x)\| &\leq \left|\int_{x_0}^{x} \|f(t,y(t))-g(t,w(t))\|dt\right| \leq \\ &\leq \left|\int_{x_0}^{x} \|f(t,y(t))-f(t,w(t))\|dt\right| + \\ &+ \left|\int_{x_0}^{x} \|f(t,w(t))-g(t,w(t))\|dt\right| \leq \\ &\leq \left|\int_{x_0}^{x} \|f(t,w(t))-g(t,w(t))\|dt\right| + \\ &\qquad + \left|\int_{x_0}^{x} L\|y(t)-w(t)\|dt\right|\end{aligned}$$

We conclude applying Gronwall's lemma.

As to the last case,we have

$$\begin{aligned}\|y(x)-u(x)\| &= \left\|\int_{x_0}^{x} f(t,y(t))dt - \int_{\xi_0}^{x} f(t,u(t))dt\right\| \leq \\ &\leq \left\|\int_{x_0}^{\xi_0} f(t,u(t))dt\right\| + \left\|\int_{x_0}^{x}[f(t,y(t))-f(t,u(t))]dt\right\| \leq \\ &\leq M|\xi_0-x_0| + \left|\int_{x_0}^{x} L\|y(t)-u(t)\|dt\right|\end{aligned}$$

We conclude applying Gronwall's lemma. □

we remark that if yz, w, v are solutions of the following Cauchy's problems

$$\begin{cases} y'(x) = f(x,y(x)) & ,x \in I_\delta \\ y(x_0) = y_0 \end{cases} \qquad \begin{cases} v'(x) = g(x,v(x)) & ,x \in I_{\delta'} \\ v(\xi_0) = v_0 \end{cases}$$

$$\begin{cases} z'(x) = f(x,z(x)) \\ z(x_0) = v_0 \end{cases} \qquad \begin{cases} w'(x) = g(x,w(x)) \\ w(x_0) = v_0 \end{cases}$$

we have

$$\|y(x)-v(x)\| \leq \|y(x)-z(x)\| + \|z(x)-w(x)\| + \|w(x)-v(x)\|$$

so we can easily find an estimate of the error committed substituting v for y. We can also state that:

Corollary 4.1 *1.* $\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0$ *such that if*

$$\|y_0 - z_0\| < \delta_\varepsilon$$

then

$$\|y(x)-z(x)\| < \varepsilon \quad \forall x \in I_\delta \cap I_{\delta_1}$$

2. $\forall\varepsilon > 0\ \exists\delta_\varepsilon > 0$ *such that if*

$$\sup\{\|f(x,y) - g(x,y)\| \ :\ (x,y) \in I \times A\ \} < \delta_\varepsilon$$

   then

$$\|y(x) - w(x)\| < \varepsilon\ \forall x \in I_\delta \cap I_{\delta_2}$$

3. $\forall\varepsilon > 0\ \exists\delta_\varepsilon > 0$ *such that if* $|\xi_0 - x_0| < \delta_\varepsilon$ *then*

$$\|y(x) - u(x)\| < \varepsilon \quad \forall x \in I_\delta \cap I_{\delta_3}$$

These results are not significant when the independent variable tends to infinity because the estimates tends to infinity too.

The behavior of the solutions at infinity It is the focus of the study of stability; it will be discussed in the next section.

4.8 Stability of solutions

Let $f : [x_0, +\infty) \times A \longrightarrow A$, $A = \{y \in \mathbb{R}^n \ :\ \|y\| < a\}$ and let us assume that

1. f is a continuous function in $[x_0, +\infty) \times A$,

2. f is locally Lipsichtz, i.e.

$$\|f(x,y_1) - f(x,y_2)\| \le L_B\|y_1 - y_2\|$$

   $\forall (x,y_1),(x,y_2) \in B$ for every bounded set $B \subset [x_0, +\infty) \times A$,

3. $f(x,0) = 0$, se $x \ge x_0$.

Then the solution of the Cauchy's problem

$$\begin{cases} y'(x) = f(x,y(x)) \\ y(x_0) = y_0 \end{cases}$$

exists and it is uniquely determined in $x \in [x_0, x_0 + \delta]$, for suitable δ; according to theorem 4.8, we assume that the solution is defined in a right maximal interval $[x_0, b)$.

Since $f(x,0) = 0$, se $x \ge x_0$, $y(x) \equiv 0$ is a constant solution of the differential equation.

In this section we always assume the preceding hypotheses.

Definition 4.5 *$y(x) \equiv 0$ is stable if $\forall\varepsilon > 0\ \exists\delta_\varepsilon > 0$ such that if $\|y_0\| < \delta_\varepsilon$, detta $y(x,y_0)$ the solution $y(x,y_0)$ of problem 4.12 is defined in $x \ge x_0$ and we have*

$$\|y(x,y_0)\| < \varepsilon \quad , \quad \forall x \ge x_0$$

Definition 4.6 *$y(x) \equiv 0$ is asymptotically stable if it is stable and moreover there exists$\delta > 0$ such that if $\|y_0\| < \delta$,*

$$\lim_{x \to +\infty} y(x, y_0) = 0.$$

A stability criterion for linear homogeneous systems can be easily proved.

Theorem 4.11 *Let $A : [x_0, +\infty) \longrightarrow \mathcal{M}^n$ be continuous and let us consider the linear differential system*

$$y'(x) = A(x)y(x).$$

Sia G a fundamental matrix for the system; then $y(x) \equiv 0$ is stable if and only if

$$\|G(x)\| \le K \ , \ \ \forall x \ge x_0$$

Moreover it is astmptotically stable if and only if

$$\lim_{x \to +\infty} \|G(x)\| = 0.$$

Proof. The solution $y(x, y_0)$ of the Cauchy's problem

$$\begin{cases} y'(x) = M(x)y(x) & , x \ge x_0 \\ y(x_0) = y_0 \end{cases}$$

can be written, by means of the fundamental matrix, in the form

$$y(x) = G(x)C$$

where $C = G^{-1}(x_0)y_0$.

Therefore we have

$$y(x, y_0) = G(x)G^{-1}(x_0)y_0$$

and

$$\|y(x, y_0)\| \le \|G(x)\| \ \|G^{-1}(x_0)\| \ \|y_0\|.$$

This estimate is sufficient to conclude both stability and asymptotic stability,

As to necessity, if $y(x) \equiv 0$ is a stable solution, then if $\|y_0\| < \delta$ we have $\|y(x, y_0)\| < 1$

Let $\{e_i \ , \ i = 1..n\}$ be an orthonormal basis for $\mathbb{R}^n$; since the solutions of the Cauchy's problem with initial data

$$y(x_0) = \delta e_i$$

form a basis for the space of the solution of the system and are bounded, we can conclude that G is bounded.

Similarly if $y(x) \equiv 0$ is a asintotically stable solution, then if $\|y_0\| < \delta$ we have $\|y(x, y_0)\| < 1$ and moreover

$$\lim_{x \to +\infty} \|y(x, y_0)\| = 0.$$

Since the solutions of the Cauchy's problem with initial data

$$y(x_0) = \delta e_i$$

form a basis for the space of the solution of the system and are bounded and infinitesimal, we can conclude that G is bounded and infinitesimal too.

□

Let us prove a stability result.

Theorem 4.12 *Let $I, R \subset \mathbb{R}$ two real intervals and let $\omega : I \times R \longrightarrow \mathbb{R}$, be a continuous function such that*

$$|\omega(x, y_1) - \omega(x, y_2)| \leq L_B |y_1 - y_2|$$

$\forall (x, y_1), (x, y_2) \in B$ *, for all bounded set $B \subset I \times R$*

Let z a differentiable function, defined in a real interval J, such that

$$\begin{cases} z'(x) \leq \omega(x, z(x)) & , x \in I_\delta \\ z(x_0) \leq y_0 \end{cases}$$

and let y the solution of the Cauchy's problem defined in a real interval J_1,

$$\begin{cases} y'(x) = \omega(x, y(x)) & , x \in J_1 \\ y(x_0) = y_0 \end{cases}$$

then

$$z(x) \leq y(x) \quad \forall x \in J \cap J_1.$$

Proof. The existence of y and z is a consequence of Peano theorem.

Let $u(x) = z(x) - y(x)$, we have $u(x_0) \leq 0$; if there was $x_1 > x_0$ such that $u(x_1) > 0$ we would have

$$u'(x) = z'(x) - y'(x) \leq \omega(x, z(x)) - \omega(x, y(x)) \leq \\ \leq |\omega(x, z(x)) - \omega(x, y(x))| \leq L|z(x) - y(x)| = L|u(x)|$$

and

$$u(x) \leq u(x_0) + \int_\xi^x L|u(t)|dt \quad , \ \forall x \in [x_0, x_1] \, .$$

By Gronwall's lemma

$$0 < u(x_1) \leq u(x_0) e^{L(x_1 - z_0)} = 0$$

which is false. □

Sometimes the stability of the solutions of a non-linear system can be reduced to the stability of the linearized system, i.e. a system obtained substituting the nonlinear part with its Taylor expansion.

This kind of study is referred to as stability in first approximation.

We can prove that if all eigenvalues the linearized system have negative real part then the original system is stable too, while the system in unstable when at least one eigenvalue has positive real part.

Theorem 4.13 *Let $P \in \mathcal{M}^n$, $A = \{y \in \mathbb{R}^n \ : \ \|y\| < a \ \}$ and*

$$f : A \longrightarrow \mathbb{R}$$

, such that $f(0) = 0$, f is a continuous function such that:

$$\lim_{x\to 0} \frac{f(x)}{\|x\|} = 0.$$

Let us consider the system

$$\begin{cases} y'(x) = Py(x) + f(y(x)) \\ y(x_0) = y_0 \end{cases}$$

We have that

- *if all the eigenvalues of P have negative real part then the zero solution is asymptoticallly stable*
- *if at least one eigenvalue has positive real part then the zero solution is instable.*

Proof.

Let us choose σ such that when $\|y\| < \sigma$ then

$$\|f(y)\| \leq \frac{\alpha}{2K}\|y\|;$$

We also assume that $a < \sigma$.

Let G the fundamental matrix of the linear system

$$y' = Py$$

such that $G(0) = I$ and let y the solution of the system defined in a maximal interval $[x_0, b)$.

Let us define

$$z(x) = G(x - x_0)y_0 + \int_{x_0}^{x} G(x - t)f(y(t))dt$$

then we have

$$\begin{aligned} z'(x) = G'(x - x_0)y_0 + \int_{x_0}^{x} G'(x - t)f(y(t))dt + G(0)f(y(x)) = \\ = P\left(G(x - x_0)y_0 + \int_{x_0}^{x} G(x - t)f(y(t))dt\right) + f(y(x)) = \\ = Pz(x) + f(y(x)) \end{aligned}$$

and

$$z(x_0) = y_0.$$

Taking in account that

$$y'(x) = Py(x) + f(y(x)) \quad , \quad y(x_0) = y_0$$

we have

$$(z-y)'(x) = P(z-y)(x) \quad , \quad (z-y)(x_0) = 0$$

whence

$$z(x) \equiv y(x).$$

Therefore

$$y(x) = G(x-x_0)y_0 + \int_{x_0}^{x} G(x-t)f(y(t))dt$$

and, since all the eigenvalues of P have negative real part, we obtain

$$\|G(x-t)\| \leq Ke^{-\alpha(x-t)} \quad , \quad x_0 \leq t \leq x.$$

We have

$$\|y(x)\| \leq Ke^{-\alpha(x-x_0)}\|y_0\| + \int_{x_0}^{x} \frac{\alpha}{2} e^{-\alpha(x-t)}\|y(t)\|dt$$

whence

$$e^{\alpha(x-x_0)}\|y(x)\| \leq K\|y_0\| + \int_{x_0}^{x} \frac{\alpha}{2} e^{\alpha(t-x_0)}\|y(t)\|dt$$

using the Gronwall's Lemma we deduce

$$e^{\alpha(x-x_0)}\|y(x)\| \leq K\|y_0\|e^{\alpha(x-x_0)/2}$$

and

$$\|y(x)\| \leq K\|y_0\|e^{-\alpha(x-x_0)} \leq K\|y_0\| \quad , \forall x \geq x_0 \tag{30.14}$$

Therefore if $0 < \varepsilon < \sigma$ and $\|y_0\| < \varepsilon/K$, we have

$$\|y(x)\| < \varepsilon \quad , \forall x \in [x_0, b)$$

The last inequality shows that y is bounded and by the prolongability theorem, we can assert that $b = +\infty$.

Taking the limit $x \to +\infty$ we can deduce the asymptotical stability.

So the first assertion is proved. the second assertion is a bit more tricky and we omit the proof. □

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