La dimostrazione ed il dimostrare

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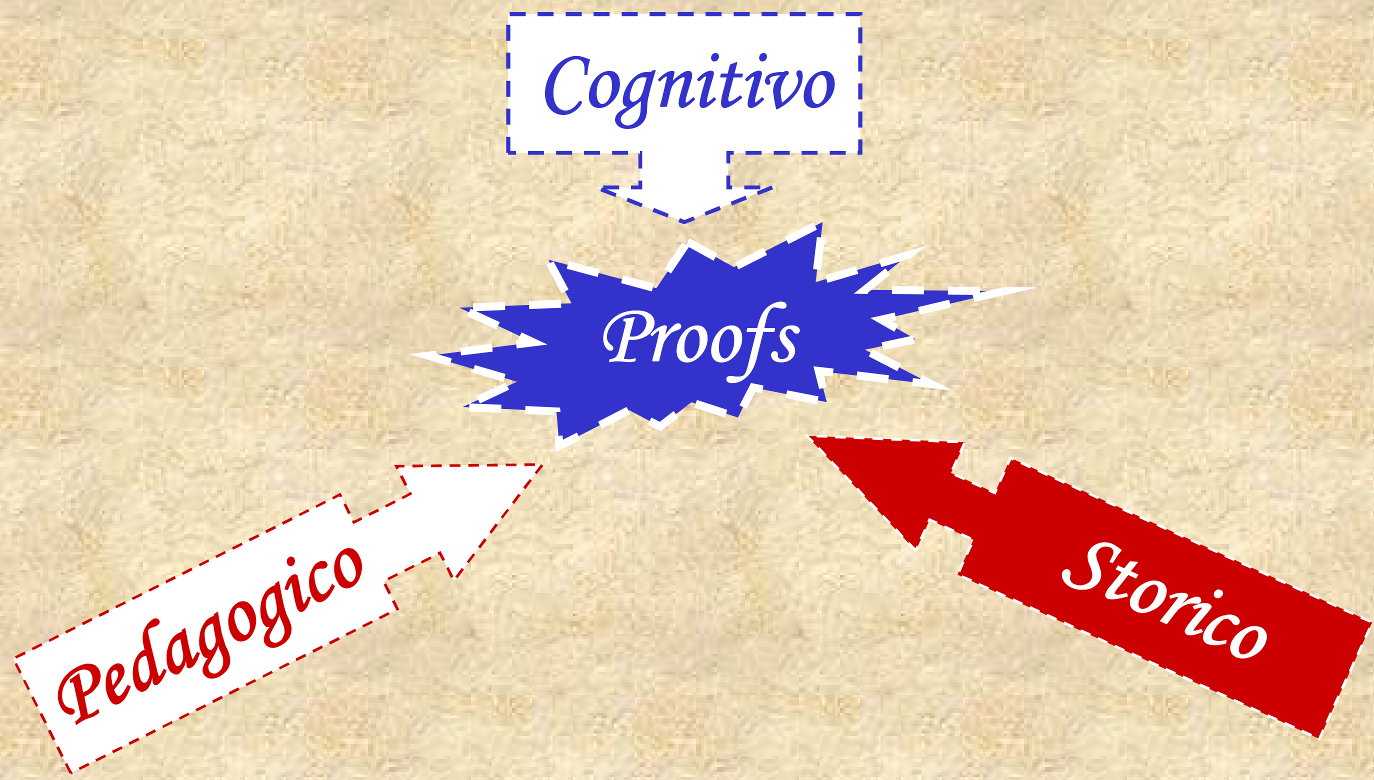
- **Cosa è una dimostrazione?**
- **Cosa è una dimostrazione matematica?**
- **Cosa significa dimostrare?**
- **Come insegniamo una dimostrazione?**
- **Quale è il ruolo di una dimostrazione?**



Prodotto

Proof as product

Insegnamento

Teaching of Proof

Processo

Proof as process

Prodotto

Proof as product

Insegnamento

Teaching of Proof

Processo

Proof as process

Prodotto

Proof as product

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Proof as process

La dimostrazione come prodotto

Precisione e rigore

Aristotele e Platone

Leibniz (1646-1716)

Frege (1848-1925)

Hilbert (1862-1943)

Prima metà del XX secolo

BOURBAKI

Il metodo matematico insegna…a trovare le idee comuni sepolte sotto l'apparato esterno dei dettagli appropriati a ciascuna delle teorie considerate, in modo da discernere queste idee ed esibirle (Bourbaki, 1971, p.26)

Anni 60

↓

DIMOSTRAZIONE FORMALE

Logicismo

Formalismo

Intuizionismo

Logicismo

La matematica è parte della logica. Lo scopo è quello di produrre un corpo per la matematica senza introdurre concetti che non siano definibili attraverso i teoremi della logica, e che non siano dimostrabili attraverso la logica proposizionale

Formalismo

La matematica è una scienza costituita da sistemi formali. Essa consiste nella manipolazione di simboli a cui non necessariamente si devono attribuire dei significati. La validità di ogni proposizione matematica risiede nell'abilità di dimostrare la sua verità attraverso dimostrazioni rigorose all'interno di un sistema formale appropriato

Intuizionismo

La matematica ed il linguaggio matematico sono due entità separate. La matematica è essenzialmente un'attività della mente priva di linguaggio. L'attività matematica dunque consiste di "costruzioni introspettive", piuttosto che di assiomi e teoremi.

La sfida alla dimostrazione formale

- **Maggiore enfasi sui contenuti di una dimostrazione e non solo sulla sua forma.**
- **Differenti livelli di validità formale, ma stesso grado di accettabilità**

Una dimostrazione formale è una sequenza finita di passi in cui il primo è un assioma, ed i successivi sono assiomi o sono stati dedotti dal passo precedente attraverso regole di inferenza; l'ultimo passo rappresenta ciò che deve essere dimostrato. Tale approccio formale fu sviluppato per eliminare la necessità dell'uso di intuizioni e giudizio umano, entrambi considerati cause di errori; infatti tale definizione elimina gli aspetti psicologici di una dimostrazione, rendendola completamente meccanica (Hanna, 1990)

La sfida alla dimostrazione formale

Sebbene la matematica non sia una scienza empirica, i suoi metodi sono molto simili a quelli delle scienze empiriche. Egli fa riferimento alla matematica come quasi-empirica. La matematica cresce attraverso un incessante processo di congetture nate dalla speculazione e da un processo critico, attraverso la logica della prova e della confutazione

Lakatos (1976)

Nuove interpretazioni

Thurston (1994)

Il congetturare è la più ovvia attività matematica che però non coinvolge il dimostrare.

"Il lavoro teorico dovrebbe essere esplicitamente riconosciuto come teorico ed incompleto; maggiore credito per il risultato finale deve essere riservato per il lavoro rigoroso che lo convalida" (p. 10)

Jaffe & Quinn (1993)

Indebolimento degli standards della dimostrazione.

Matematica **Teorica** (Theoretical mathematics)

Fase speculativa, costruzione di congetture

Matematica **Rigorosa** (Rigorous Mathematics)

Fase in cui le congetture e le speculazioni vengono corrette; e vengono rese attendibili attraverso la loro dimostrazione

Thurston (1994)

In che modo i matematici aumentano la comprensione della Matematica?

La misura del nostro (matematici) successo è dovuta alla nostra capacità di mettere in condizioni le persone di capire e pensare più chiaramente ed in maniera effettiva la matematica (p.163)

L'importanza della comprensione e dei modi di pensare (ways of thinking)

I matematici dovrebbero porre maggiore sforzo nel comunicare idee di tipo matematico, e nel fare ciò si dovrebbe porre molta più attenzione nel trasferire non solo definizioni, teoremi e dimostrazioni, ma anche ***modi di pensare.*** *Vi è la necessità di apprezzare il valore dei differenti modi di pensare riguardo alla stessa struttura matematica. Matematici e ricercatori in didattica della matematica devono focalizzare maggior attenzione sull'apprendimento e sul modo di spiegare le infrastrutture mentali di base della matematica (Thurston, 1994)*

Thurston vs. Jaffe & Quinn

La distinzione operata da Jaffe & Quinn non è altro che un'ulteriore strumento per perpetuare il mito che il nostro successo viene misurato in base agli standard deduttivi utilizzati per provare un teorema

Nuove interpretazioni

Computer Era

- Il teorema dei quattro colori (Appel and Hanken) The four-color theorem
- La soluzione al "party problem" (Radziszowski and Mac Kay)

- **Zero-Knowledge proof** (Blum, 1996)
- **Holographic Proof** (Cipra, 1993; Babai, 1994)

Computer Era

Goldwasser, Micali, Rackoff (1985).

Protocollo Interattivo che coinvolge un prover e un verifier

- Il teorema dei q

he four-color theorem

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Forma trasparente.

Verifica attraverso spot checks

Diversi quesiti

- **Are such proofs going to be the way of the future?**
- **Do such proofs have a place in mathematics? Are we even allowed to call them proofs?**
- **Should mathematicians accept mathematical propositions which are only high probably true as the equivalent of propositions which are true in the usual sense?**
- **If not, what is their status?**
- **Should mathematicians accept proofs that cannot be verified by others, or proofs that can be verified only statistically?**
- **Can mathematical truths be established by computer graphics and other forms of experimentation?**
- **Where should mathematicians draw the line between experimentation and deductive methods?**

(Horgan, 1993; Krantz, 1994; Babai, 1994)

Il Ruolo della Dimostrazione

Davis
(1986)

Il Ruolo della Dimostrazione

Davis (1986)

validare

Il Ruolo della Dimostrazione

Il Ruolo della Dimostrazione

Il Ruolo della Dimostrazione

Davis (1986)

Eliminare errori

Oggetto di dibattito

validare

Portare a nuove scoperte

Il Ruolo della Dimostrazione

Hanna (1990)

Hanna (1990)

Dimostrazioni che dimostrano

Proofs that prove

Il Ruolo della Dimostrazione

Hanna (1990)

Dimostrazioni che dimostrano

Proofs that prove

Dimostrazioni che spiegano

Proofs that explain

Confronto tra una dimostrazione che dimostra ed una che spiega

Prove that the sum of the first n positive integers, S(n), is equal to n(n+1)/2

A proof that proves

Proof by mathematical induction:

For n=1 the theorem is true.

Assume it is true for any arbitrary k.

Then consider:

S(k+1)= S(k)+(k+1)=)=n(n+1)/2+(n+1)= (n+1)(n+2)/2

Therefore the statement is true for k+1 if it is true for k.

By the induction theorem, the statement is true for all n.

Now, this is certainly an acceptable proof: it demonstrates that a mathematical statement is true. What it does not do, however, is show *why* the sum of the first n integers is n(n+1)/2 or what characteristic property of the sum of the first n integers might be responsible for the value n(n+1)/2. (Proofs by mathematical induction are non-explanatory in general).

Gauss's proof of the same statement, however, is explanatory because it uses the property of symmetry (of two different representation of the sum) to show why the statement is true. It makes explicit reference to the symmetry, and it is evident from the proof that its result depends on this property:

A proof that explains

Gauss's proof is as follows:

S= 1+ 2 + 3 +…….+n

S=n+ (n-1)+ (n-2) +…….+1

2S=(n+1)+(n+1)+(n+1)+……+(n+1)=n(n+1)

S=n(n+1)/2

Another explanatory proof of this same statement is, of course, the geometric representation of the first n integers by an isosceles right triangle of dots; here the characteristic property is the geometrical pattern that compels the truth of the statement. We can represent the sum of the first n integers as triangular numbers (see Figure 1)

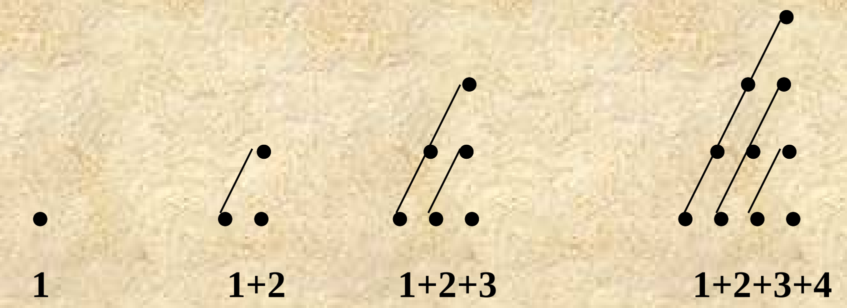


Figure1

The dots form isosceles right triangles containing

S(n)=1+2+3+….+n dots

Two such sums S(n)+S(n) give a square containing n^2 dots and n additional dots because the diagonal of n dots is counted twice. Therefore:

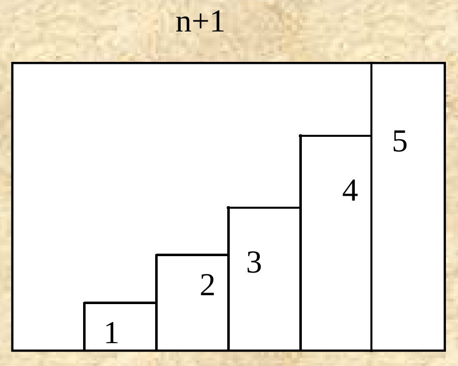
$$2S(n)=n^2+n$$

$$S(n)= (n^2 +n)/2=n(n+1)/2$$

Another explanatory proof would be the representation of the first n integers by a staircase-shaped area as follows: a rectangle with sides n and n+1 is divided by a zigzag line (see figure 2).

Another explanatory proof would be the representation of the first n integers by a staircase-shaped area as follows: a rectangle with sides n and n+1 is divided by a zigzag line (see figure 2).

The whole area is n(n+1), and the staircase-shaped area, 1+2+3+…+n only half, hence , n(n+1)/2



Both Gauss's proof and the geometric representation show that one can adopt an explanatory approach to proof in the classroom without abandoning the criteria of legitimate mathematical proof and reverting to reliance on intuition alone. What one must do, rather is to replace one proof, of the non-explanatory kind, by another equally legitimate proof which has explanatory power, the power to bring out the mathematical message in the theorem (p. 10-11).

Dimostrazione ed Insegnamento

New Math (anni 50)

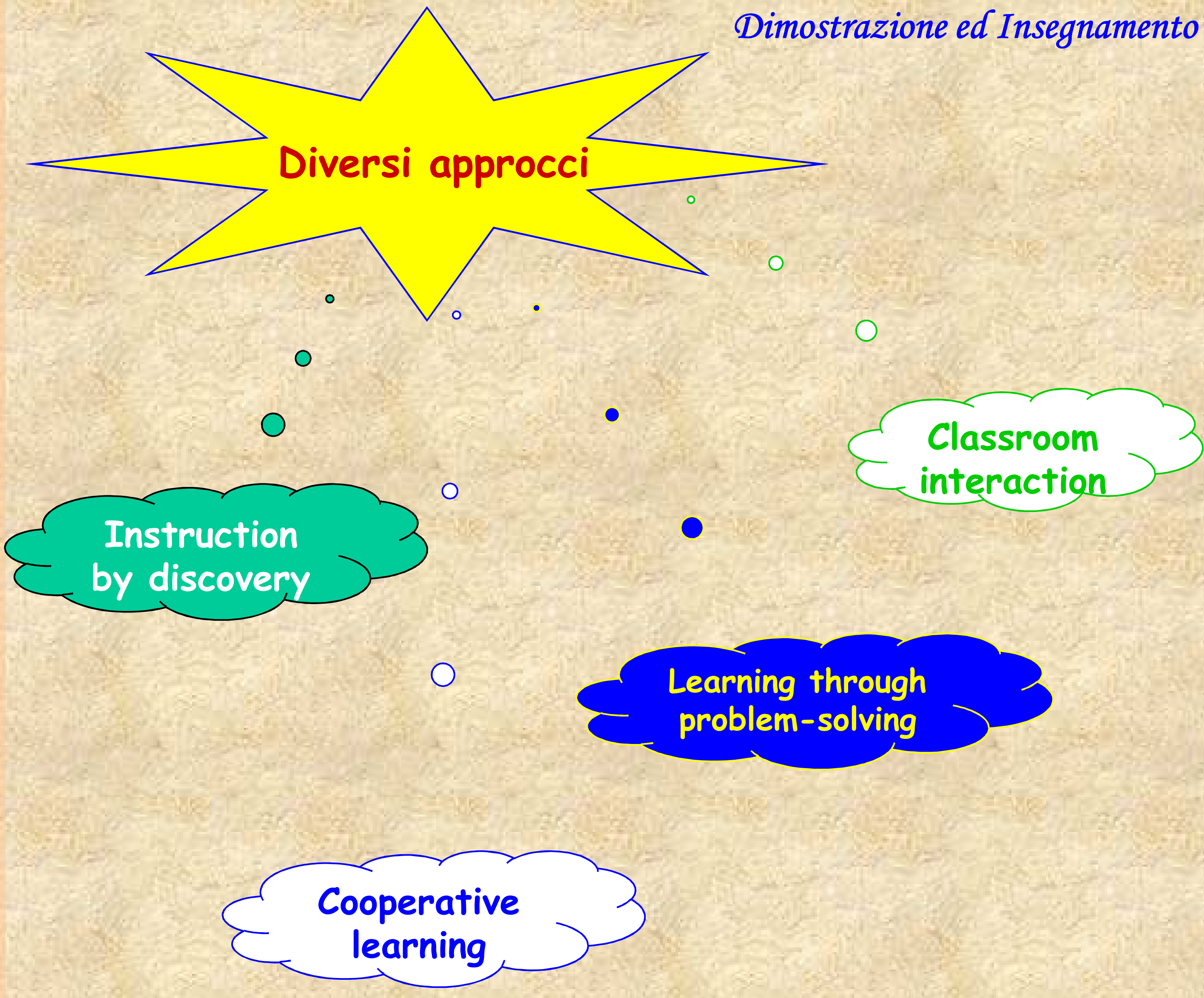
Esagerata enfasi alla dimostrazione formale

1. Introduzione

2. La dimostrazione come prodotto

3. Dimostrazione ed insegnamento

Dimostrazione ed Insegnamento



Constructivism

Von Glasersfeld, 1983; Cobb, 1988; Kieren and Steffe, 1994

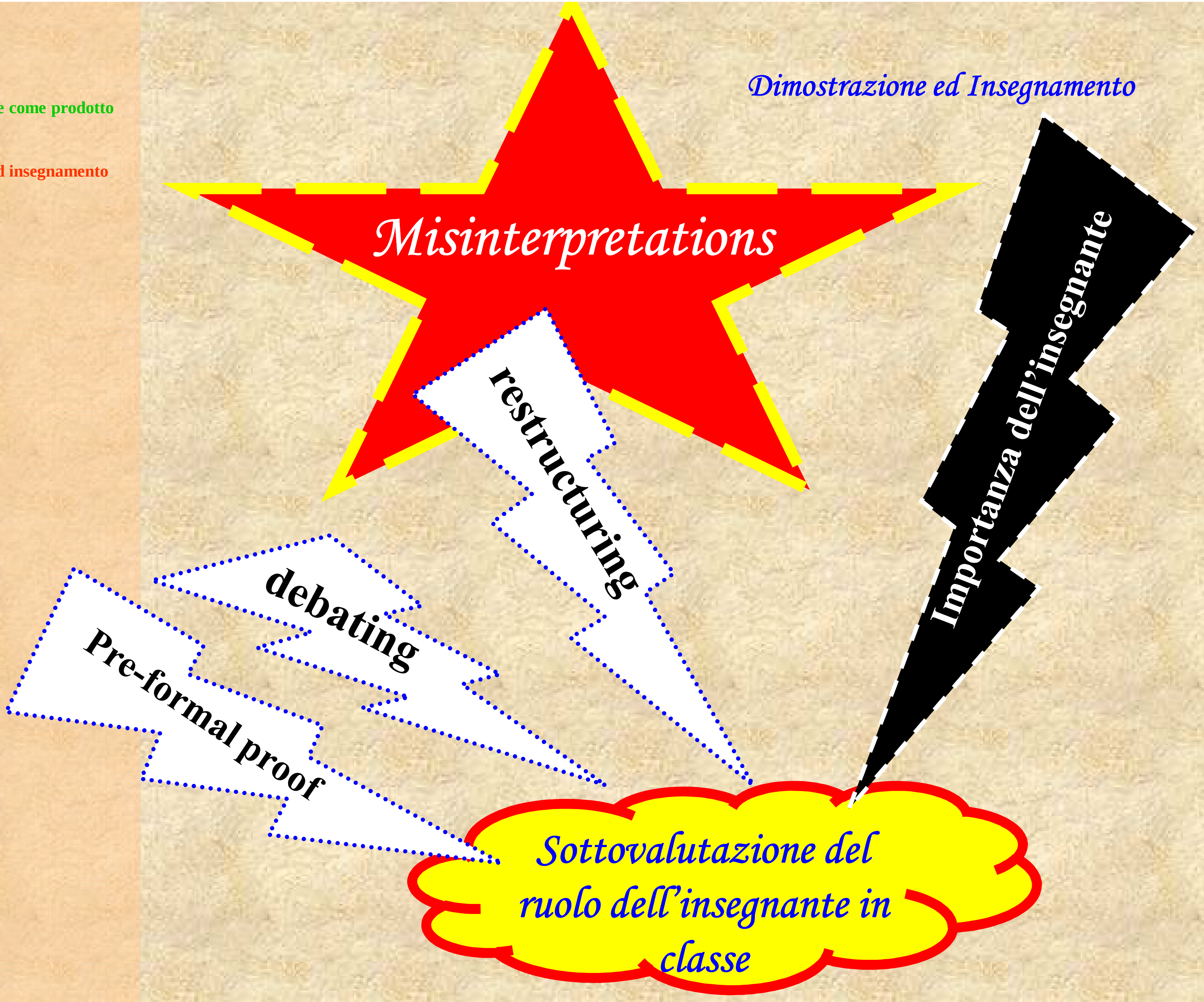
La conoscenza viene costruita da colui che impara

1.Introduzione

2. La dimostrazione come prodotto

3. Dimostrazione ed insegnamento

Dimostrazione ed Insegnamento



Classroom debate

Meanings are not used as means for the control of the results of an algorithm

Alibert, 1988

Experimental teaching method

- **Théorie des situations didactique (Brousseau, 1986)**
- **Plurality of the conceptual settings (Douady, 1986)**
- **Need of proof generated by the contradictions (Balacheff, 1982)**
- **Importance of the group for the construction of meaning (Bishop, 1985; Balacheff & Laborde, 1985)**
- **Meta-mathematical factors (Schoenfeld, 1983)**
- **Learner's epistemology**

Primo passo: organizzazione da parte dell'insegnante di costruzione di statements scientifici da parte degli studenti

Secondo passo: Lo statement è rimesso agli studenti per considerazioni e discussioni. Arrivano alla decisione della validità attraverso il voto; ognuno deve supportare in qualche modo , attraverso un argomento scientifico, una dimostrazione, un contro-esempio…

Terzo passo: statement validato da dimostrazione diventa teorema. Altrimenti conservato come statement falso

Dimostrazione per convincere

Proofs to convince

Dimostrazione per mostrare

Proofs to show

"Dibattito scientifico"

Dimostrazione per convincere

Proofs to convince

Per convincere qualcuno (un altro studente)
Non parte di conoscenza istituzionalizzata

"Dibattito scientifico"

Dimostrazione per mostrare

Proofs to show

Per convincere qualcuno (l'insegnante)
Raggiungimento di una conoscenza dall'altro già posseduta

"Dibattito scientifico"

Uno studio sperimentale

The purpose of the questionnaire is to investigate which kind of "culture of proof" students own; and what conceptions and misconceptions they have about this issue.

What is the relationship between students' own view of themselves regarding to:

a) their ability to produce proofs;

b) their perceived freedom to produce proof;

and to their ability to reason abductively?

Il questionario

Note: For the following question it is possible to choose more than one answer.

1. CHECK THE FORMS OF REASONING YOU KNOW

❑ Induction

❑ Deduction

❑ Others. Which ones?

2. AS STUDENT, DO YOU THINK THE STUDY OF PROOFS TO BE NECESSARY?

❑ Yes. Why?

q No. Why?

Sometimes. When?

1.Introduzione

2. La dimostrazione come prodotto

2.1 Nuove interpretazioni

2.2 Il ruolo della dimostrazione

3. Dimostrazione ed insegnamento

3. WHICH KIND OF RELATIONSHIP LIES BETWEEN HYPOTHESIS AND THESIS IN THE CONSTRUCTION OF THE STATEMENT OF A THEOREM?

The hypothesis comes always before the thesis. Why?

The thesis comes always before the hypothesis. Why?

Depends (Justify it)

4. FOR EACH THEOREM DO YOU THINK THAT THERE EXISTS ONLY ONE CORRECT PROOF?

Yes. Why?

No. Why?

5. THE CONSTRUCTION OF A PROOF HAS TO FOLLOW A FIXED PATTERN. CREATIVITY CANNOT FIND ROOM IN THE COSTUCTION OF PROOFS.

True. Why?

False. Why?

Note: for the following question it is possible to choose more than one answer

6. A PROOF IN CALCULUS HAS THE FOLLOWING ROLE

Convince someone of the validity of a statement

Explain why a statement is valid

Establish the validity of a statement

Other (specify)

Il questionario

1. CHECK THE FORMS OF REASONING YOU KNOW

- ❑ Induction
- ❑ Deduction
- ❑ Others. Which ones?

just a survey tool to check which forms of reasoning students know, if they define or recognize as forms of reasoning others than the inductive and deductive ones.

Il questionario

2. AS STUDENT, DO YOU THINK THE STUDY OF PROOFS TO BE NECESSARY?

❑ Yes. Why?

q No. Why?

Sometimes. When?

wants to investigate with what kind of "mental attitude" students approach a proof. If they tackle the construction of a proof just because they are said so by the teacher, or if there is a sort of curiosity about that, and a conviction about its necessity. The "why" question is to browse also what kind of influence school could have had in students' opinion about such an issue.

3. WHICH KIND OF RELATIONSHIP LIES BETWEEN HYPO… IS AND THESIS IN THE CONSTRUCTION OF THE STATE… OF A THEOREM?

The h… is comes always before the thesis. Why?

The thesis … Why?

to discover and analyze students' conception about the structure of a proof. Very often students are involved in dealing with "ready made" proofs. Which kinds of "cognitive processes" and "back and forwards" reasoning, not always so linear and "monotonic" (see Magnani, Mason et al.), have been made by the mathematician who produced such a proof is very often an alien topic for the student himself. This means that unfortunately students very seldom have the opportunity to deal with a "proof in progress", on the contrary they just have experience with a kind of didactical contract that looks at the teacher as the only source of truth who just transfers some pre-constructed knowledge from himself to the class. (an authoritarian proof scheme, see Harel)

4. FOR EACH THEOREM DO YOU THINK THAT THERE EXISTS ONLY ONE CORRECT PROOF?

Yes. Why?

No.

to understand the idea, regarding proofs, students have built during their scholastic career. If they think it is possible any theorem may have just one correct proof, or if they don't relate creativity and personal initiative with the process of construction of a proof, because they just experienced during the years the final product of it. If so, we may interpret their difficulty in the approach of proving process and their reluctance to tackle an open problem because they just wait for somebody tells them how to proceed.

Note: for the following question it is possible to choose more than one answer

6. A PROOF IN CALCULUS HAS THE FOLLOWING ROLE

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Other (specify)

The final question is very important, namely, it is really fundamental to try to understand which role students give to a proof, because it is such an idea that leads their predisposition to the construction of it.

Alcuni risultati

89 studenti (I anno università)

Q1: The majority of students (52%) knows both Induction and Deduction; followed by students who know not only Induction and Deduction but also proof by contradiction.

Q2: 58/89 answered **Yes**, 5/89 answered **No**, 26/89 answered **Sometimes**.

Most of the students (65%) thinks of proofs as a tool to understand better theorems, their meaning, and the reasoning involved into the process of proving. The remaining part is mainly concerned with the idea that proofs are necessary because they validate the problem and convince of its validity, or as a tool useful to solve problems, to create mental schemes to be used in problem-solving, furthermore they explain the why of a fact, and finally they make a context clearer, and easier to be remembered.

Most of the students who answered “Sometimes” (29%) states that proof is necessary when it helps to understand better a theorem.

Very few (6%) are convinced that proofs are not necessary at all.

Conclusion: the main idea about the necessity of a proof is based on its usefulness to help oneself to understand better a theorem; therefore proof is seen as an ***explanatory*** tool.

Q3: 32/89 answered **It depends**, 12/89 answered **The thesis comes always before the hypothesis**, 45/89 answered **The hypothesis comes always before the thesis.**

Concerning the first choice (it depends) we could summarize the main justifications as follow: very often the thesis is considered as a starting point from which it is possible to build the hypotheses that may prove the validity of the thesis itself; not only but the thesis seems to own an empirical connotation in contraposition with a more cognitive connotation of the hypothesis; namely, the thesis comes from an observed fact, while hypothesis is the construction of a reasoning.

Very interesting is the answer given by a student who reveal the awareness of the difference between the construction of a proof and its “formalization”. Moreover, part of the students relate the characteristics of a proof with the cultural background.

Among the 32 students (36%) who answer "It depends", almost half of them (15/32) seems to base their response "the thesis comes before the hypothesis" on a common idea: the experimental characteristic of the reality; that means: in the real world what is observed is a fact (the thesis) that may be unusual or at least not directly explainable, therefore we look for or we try to build some hypothesis which may justify, or validate the observation that has been made. Such students seem to describe the process Peirce talks about regarding *abduction*. Below the most significant answers given by the students to this regard are listed.

1. Many times you start from the thesis and then you build the hypothesis useful to prove the validity of the thesis itself

2. I think that there is a difference between the moment you state a theorem (usually the hypotheses are listed in an orderly way, then the thesis go after) and the construction of the statement of a theorem. This one follows a very laborious and "untidy" process; to this extent, sometimes you may have in your mind a result and you need to look for hypotheses from which you obtain the result; other times you start from certain hypotheses and you try to understand what they lead to. Besides, in the famous "if and only if" hypotheses and theses exchange the role.

3. Many times you know where you want to arrive, but you don't know where to start from.

4. Sometimes it happens that you have an intuition on a thesis and subsequently you build the hypotheses that make the thesis true.

5. Sometimes the thesis is already known and the proof is used only to explain the why of the validity of the thesis.

6. It depends, because often a theorem rises from empirical experience, and therefore the thesis comes before the hypothesis.

7. It depends if you infer the thesis from a group of hypothesis (a trivial and not very useful case) or if you need a thesis, or if you want to verify it, and you look for the hypotheses which you infer the thesis from (much more common case).

The second choice is given by "the thesis comes always before the hypothesis". Again, the general idea supporting this answer is that the thesis is the fact, the problem to be solved, the starting point, and the hypothesis is the tool to explain, to validate the observed fact. A new idea seems to come out from students' justifications, it is the sequence between the hypothesis and the thesis. Namely, the existence of a hypothesis is subordinate to the presence of a thesis as some students wrote:

1. First I decide what has to be proved

2. Anybody, before of choosing to use particular tools and conditions (hypotheses) to prove his/her own conviction, has to have first a conviction that his/her own genius judges to be correct

3. The hypotheses of a proof are built afterwards, because they put some "limits" (they are "characteristics") for the statement of the theorem.

The last choice was represented by: "*the hypothesis comes always before the thesis*". In this case hypothesis, for example, is considered like what you already know and thesis is the unknown, therefore we start from what we know to prove the thesis.

Another interpretation is given by "the hypothesis is the place where the data necessary for the proof lay "; other times the relationship between hypothesis and thesis seems to be the same of the logical sequence "first doubt and then certainty". To this extent the hypothesis represents the doubt and the thesis is the certainty.

The majority of the students seems to be influenced by the structure (and not by the creation) of a proof as it is usually presented at school; therefore, proof is just a a sequence of steps, that start from hypotheses to end into a thesis. Such a rigid structure is so predominant the student doesn't realize that himself assumes the presence of a thesis before the statement of the hypotheses ("the supposed thesis"). Nevertheless, a presence of a thesis before the hypotheses seems not to be part of the process of proving.

Furthermore, hypotheses seem to live of their own life; the thesis rises as a consequence of the reasoning made about the hypotheses. But why such hypotheses are made or taken on consideration we don't know…

The last interpretation of hypothesis I am going to take on consideration is the most interesting. Several students identify *hypothesis* only with *supposition, conjecture*, and look at the thesis as a hypothesis whose true value has been proved; namely, a thesis is a previous hypothesis (conjecture) that has been proved to be true. Therefore, hypothesis and thesis are the same statement with two different value of truth: till when the statement is not proved to be true, it is a hypothesis, after its proof of true value it becomes a thesis. On the contrary hypotheses meant as a set of rules, axioms etc…already true, are not considered as hypotheses but just a set of statements.

Below the most significant excerpts has been taken on consideration to underline the explanations given by the students.

1. From known things you prove unknown things

2. Without hypotheses you can't arrive at any thesis

3. The hypothesis gives the basis in order to prove the thesis and for its proof, therefore it is essential

4. From the hypothesis or hypotheses applying mental-logical steps or theorems already known and proved, or axioms, you arrive always at the thesis no matter complicated the theorem is.

5. Because first you state some hypotheses and then you try to reach the supposed thesis

6. Because you always make a hypothesis first and then after several proofs you may give a thesis

7. First I state the hypothesis and from that I reason to state my thesis

8. Because I suppose a fact and then I prove that it is true

9. Because you suppose a hypothesis to be true, and through a set of statements, you reach a thesis

Q4: Almost the totality of the students agree with the fact that there may exist more than one correct proof for the same theorem.

Different are the justifications given by the students. Some of them seem to be influenced by their scholastic experience, in the sense that they legitimate the existence of more than one correct proof, because they saw it at school (a sort of authoritarian scheme).

Others start from the idea that existing different ways of reasoning and different tools (axioms, postulates, and so on), there must exist different ways to make a proof for the same statement

Furthermore, a proving process depends on our own knowledge, for this reason such a procedure may take different aspects, not only but also, different levels of knowledge lead to different levels of proof. Interesting is the fact that students seem to be aware of the existence of several correct proofs for the same theorem, but they meet just one of them during their scholastic career.

The sentence "only one is taught" underlines the passive character of the students' learning process; usually proofs are presented to students as a ready made product, instead to be involved actively in the construction of it.

1. Many times during high school I saw theorems proved in different ways but all correct

2. Proof is strictly dependent on the kind of reasoning you made.

3. Because many times it is possible to take different ways to prove something. All depends on the knowledge a person has and also on the ways he/she has been taught to reason.

4. I think there are theorems which have more than one correct proof, because these proofs can be built using different mathematical tools, sometimes more sophisticated , sometimes less, but also because they are situated in different mathematical contexts. (You may find a theorem both in analysis and in geometry for example). This is the reason why the same theorem may have a two lines proof and another may have a two pages proof.

5. It depends on your knowledge background, a competent person may proof a theorem in a complicate way, for example with more advanced knowledge, but sometimes you may prove a theorem with easier tools [...]

6. You may use several methods to make a proof; you may start from different points of views and reach the same thing. This depends on the knowledge and on the tools you have, and furthermore it depends also on what view point you want to prove (e.g., mathematical, physics)

7. In my opinion it is possible to reach a proof following different ways, sometimes there doesn't exist a correct proof but there may exist several correct proofs

8. A proof may follow different paths depending on the kind of study and level of knowledge [...]

9. I think there exist several ways to proof a theorem; but usually only one is taught.

Q5: 80 students out of 89 claim that creativity and personal initiative are fundamental parts of a proving process. Many different justifications have been given to explain such a choice. For example, creativity and personal initiative are fundamental but are acceptable only when they respect a sort of rigor, peculiar characteristic of mathematics science; furthermore, creativity can be taken on consideration when is based on a cultural background and on recognized knowledge. Always in this case they underline that the limitations about the rigor are not related to the proof's structure but to the concept to be used.

For other students creativity and personal initiative are considered as a smart mind's characteristic.

Furthermore, the aforementioned skills have to be considered possible tools of a proving process, because there is no only one way to approach a proof, namely among the different ways to tackle a proof there is intuition and creativity.

Nevertheless, there are two different explanations about this issue; some students justify the use of creativity and intuition arguing that they are just one of the several methods which can be used; others state that there may exist several ways to approach a proof because of creativity and intuition.

Students consider that many times proofs are very difficult; creativity and intuition may help to approach such a process in a easier way, not only but they enhance scientific progress and new knowledge. A possible explanation might be that students recall their scholastic knowledge about great philosophers, mathematicians and tinkers of the history.

An important tool that allows to look at the problem from different points of view; to reason at "360 degrees", to look beyond what the "eyes of mind" may see. Prefixed rules may become a cognitive obstacle that may be overcome by intuition or personal creativity, that also enhance the development of sense of critique. Therefore, creativity and intuition as an instrument of exploration, of construction of new knowledge, it is considered as a "reading key".

Furthermore, creativity and intuition are necessary to enhance fantasy; for some students fantasy is an important component in the process of proving, because many proofs are very artificial, and in order to find such artifices you need a lot of fantasy. Probably students think of proofs like the one for Lagrange theorem, Taylor theorem, or the first derivative of the product and so on.

Finally, there are no fixed schemes, any problem is different to another one, for this reason we have to employ creativity and personal initiative. In addition, no machine may build any kind of proof, therefore creativity is needed. To conclude, creativity and personal initiative are the tools to communicate with the others, and to make oneself understand.

The remaining students, exactly 9, argue that creativity and personal initiative cannot be part of a proving process. First of all because mathematics is an applied science and the previous two cannot be applied; mathematics is a universal science that must use universal tools, understandable by everybody, and creativity cannot be considered universal, on the contrary it is subjective.

Furthermore, in mathematics there are fixed rules that cannot be changed. For some students creativity is part of the formulation process of a hypothesis, therefore it cannot be part of a proving process. It seems that the formulation of a hypothesis and the process of proving be two disconnected things.

Finally, mathematics is seen as a whole of fixed rules and schemes that must be followed with rigor.

1. Because, if you mean creativity in the sense of freedom to start from where you want, I think it is possible to do it, what it is important is to be able to prove what you want. Probably, the limitations are not much in the structure of the proof but in the concepts you may use. A rigorous proof uses abstract concepts because stillness, invariability in time of the proof must be guaranteed

2. Creativity in mathematics is the most difficult thing, but also the most beautiful (if correct). It may simplify steps that are only mechanics therefore boring. What is fundamental, anyway, is the fact that mathematical rules have to be respected.

3. There are many ways to prove a theorem. Therefore personal initiative and creativity are at the basis of a proof

4. Even the history teaches us: "a spot of genius" may lead to a proof that is totally out of traditional schemes adopted to build a proof

5. Many times without intuition, creativity, and personal initiative you cannot find an efficient proof

6. I think that creativity and personal initiative are the most important tools in the construction of a proof, because they help to think of and to wonder about problems of different kind (even though later on some of them may result not useful) and creativity and personal initiative develop a capacity' of personal critical analysis

7. Creativity and personal initiative may lead to the discovery of alternative proofs sometimes correct, sometimes not. Anyway, such proofs may be useful to shed light on some properties not yet found

8. It is exactly creativity that makes us to think at 360 degrees, and to explore several ways and methods for a proof

9. It is thanks of famous mathematicians' creativity that many theorems have been discovered. Following fixed schemes cannot be enough, because sometimes you have the solution in front of your eyes but you cannot see it with the eyes of the mind.

10. Theorems and axioms must be "fixed", but often it is intuition deriving from personal initiative that leads to the construction of a correct proof

11. Being any problem different from the others, it would be wrong to think to solve it adopting procedures that follow a universal scheme.

12. There are some rules that has to be followed

13. A proof is a mathematical procedure that doesn't leave space to conjectures or creativity in the sense that any employed procedure must follow laws that are in a certain way and that cannot be in any other way. All you use for a proof is regulated by mathematical laws

Very often in going through a formal proof , particularly those suffering from the"let us define" syndrome (Avital, 1973) the student feels treated shabbily. The origin of the proof remains a mystery and the student is left with a frustrated feeling of not being wise enough not only not as wise as the person who invented the proof, but not even wise enough to understand how the inventor came up with the idea.

The attitude towards mathematics which is encouraged this way is:

"I'll never understand it, it is not for me".

(Movshovitz-Hadar, 1988)