ABDUCTIVE PROCESSES

IN CONJECTURING AND PROVING

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To my father, Walter, who is always in my heart, and my mother Wanda.

"True Science Discovers God
Waiting Behind Every Door"

Pope Pius XII

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ABSTRACT

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The purpose of the present study was to build a cognitive model to identify and account for possible cognitive processes students implement when they prove assertions in Calculus, specifically a cognitive model that would help to recognize creative processes of an abductive nature. To this end, Peirce's *Theory of Abduction* and Harel's *Theory of Transformational Proof Scheme* have been used. The result has been the construction of the Abductive System whose elements are {facts, conjectures, statements, actions}; briefly, *conjectures* and *facts* are 'act of reasoning' generated by phenomenic or abductive actions, and expressed by 'act of speech' which are the statements. At the base of the construction of the Abductive System there is also the intention to show that the creative processes own some components, and to separate this process from the belief that it is not possible to talk about it because it is something indefinable and only comparable to a "flash of genius". The common denominator with Peirce's work is the philosophic spirit on which both works are based. Peirce wanted to legitimate the fact that abduction is a kind of reasoning along with deduction and induction, in contrast with many philosophers who regard the discovery of new ideas as mere guesswork, chance, insight, hunch or some mental jump of the scientist that is only open to historical, psychological, or sociological investigation. The definition of Abductive System allows the researcher to analyze a broader spectrum of creative processes, and it gives the opportunity to name and recognize the abductive creative components present in the protocols. From the didactical point of view, it allows to recognize the variety of the components of the creative processes, in order to respect them (usually it is not done this

way at school) and to improve them. Therefore, this framework could help teachers to be more conscious of what has to be 1) recognized, 2) respected, and 3) improved, with respect to a didactic culture of “certainty”, which follows preestablished schemes.

1. INTRODUCTION

Research in mathematics has long acknowledged the importance of autonomous cognitive activity in mathematics learning, with particular emphasis on the learner's ability to initiate and sustain productive patterns of reasoning in problem solving situations. Nevertheless, most accounts of problem solving performance have been explained in terms of inductive and deductive reasoning, paying little attention to those novel actions solvers often perform prior to their engagement in the actual justification process. For example, cognitive models of problem solving seldom address the solver's idiosyncratic activities such as: the generation of novel hypotheses, intuitions, and conjectures, even though these processes are seen as crucial steps through which mathematicians ply their craft (Anderson, 1995; Burton, 1984; Mason, 1995).

The purpose of this study is to build a cognitive model to identify and account for the possible cognitive processes students implement when they perform conjectures and proofs in Calculus, and, more specifically a cognitive model that will help to recognize creative processes of an abductive nature. To this end, Peirce's *Theory of Abduction* and Harel's *Theory of Transformational Proof Scheme* are used.

The questions leading the research are:

1. Are the definitions of abduction, already given, sufficient to describe creative processes of abductive nature? Or, is a broader definition of abductive process needed to describe some creative students' processes in mathematics proving? If so, what is that definition?

2. Is one's certainty about the truth of an assumption an indication for an initiation of abductive reasoning in her or his process? Namely, how much is important the level of confidence of the built answer to guide an abductive approach?

3. Is there continuity between the cognitive “tools” one uses to build a conjecture and the means one uses to establish its validity?
4. Which elements convey an abductive process? In particular, does transformational reasoning facilitate an abductive process?

Chapter 2 presents the literature this research is based on. The reader will find four major tenets: the first tenet is related to the definition of abduction and its role, considered under three different points of view: a) the logical and philosophical point of view (Charles S. Peirce); b) the solving-problem process point of view (Cifarelli); c) the adoption of the definition of abduction in different contexts (Magnani). The second tenet concerns the Theory of Proof Schemes (Harel, 1998) and, particularly, the role of the Transformational Proof Scheme and Harel’s definition of proof scheme for a subject. The third tenet involves the “Reference System Continuity” (Garuti, Boero & Mariotti, 1996; Pedemonte, 2002) born as a product of a study concerning the difficulties met by the students in the approach to proof. The last part of the chapter deals with the topic of proofs considered in three different conditions, namely, a) proof as product; b) proof as process; c) the teaching of proof.

Chapter 3 presents the core of the research. Specifically, it describes the construction of the *Abductive System*, which has been created with the aim to give new tools to identify and analyze creative abductive processes involved when the subject is faced with a task in Calculus.

Chapter 4 deals with the methodology, the reader will find the description of the site and the participants, and how the data were collected and analyzed, the text of the two exercises given to the students who participated at the research project, and the text of a questionnaire given to the students with the aim of understanding their ideas about the meaning and the role of a proof in mathematics.

Chapter 5 presents the analysis of the students’ protocols. The analysis of the protocols is divided into two phases. The first phase shows a comprehensive description of students’ behaviors in tackling the problem; in the second phase the creative processes are detected and interpreted through the elements of the abductive system.

Chapter 6 deals with the discussion of the results brought to light by the previous analysis. The chapter is divided into four sections. The first section evidences the importance of creative abductive processes in mathematics, but more generally in the sciences. The following three sections are dedicated to the analysis of three different conditions, which seem to enhance the manifestation of creative abductive processes. Briefly, these conditions are:

1. A didactical contract that encourages and emphasizes creative processes aimed at understanding *how things work* in mathematics (section 2).
2. The chance of favoring (with an appropriate choice of tasks) transformational and perceptual reasoning (Harel, 1998) to pass from the phase of exploration to the phase of creative abductive act of reasoning (section 3).
3. The chance of favoring (with an appropriate choice of tasks) the "reference system continuity" between the conjecturing phase and the evidencing phase, as a facilitating condition for the success of the student, and therefore of his or her satisfaction to fulfill the requirement of the task (section 4).

Section 5 discusses the kind of experimental sample taken into consideration, which is represented by a group of students who voluntarily agreed to participate in the research for this project.

Chapter 7 proposes the conclusions of my research and some implications for further research.

In the last part of the thesis the reader will find the complete student transcripts, the scanned samples of their protocols, and the data analysis of the questionnaire given to the students at the beginning of the research project.

2. LITERATURE REVIEW

2.1 Peirce and his theory of Abduction

The majority of philosophers deny there is any logic in proposing a hypothesis. For them the logic of discovery (if it can be properly called such) can only be concerned with the investigation of the methods of testing hypotheses, which have already been presented to us. Popper argues, “The initial stage, the act of conceiving or inventing a theory, seems to me neither to call for logical analysis nor to be susceptible of it. The question how it happens that a new idea occurs to a man…may be of great interest to empirical psychology; but it is irrelevant to the logical analysis of scientific knowledge”(Popper, 1959).

Thus, some philosophers have come to regard the process of constructing and selecting a hypothesis as a reasonable affair, which is susceptible to logical analysis. They feel that in scientific discovery, there may be more problems for the logician than simply analyzing the arguments supporting already invented hypotheses. Peirce writes, “each chief step in science has been a lesson in logic”(5.363). He apparently feels that there is a *conceptual* inquiry, one properly called “a logic of discovery,” which is not to be confounded with the psychology, sociology and history of discovery. However, most contemporary philosophers are unreceptive to this view, giving most of their attention to inductive reasoning, probability, and the principle of theory construction. Hanson, a staunch supporter of Peirce’s view, writes “But for Peirce, the work of Popper, Reichenbach, and Braithwaite would read less like a *Logic of Discovery* than like a *Logic of the Finished Research Report.* Contemporary logicians of science have described how one sets out reasons in support of a hypothesis once proposed” (Hanson, 1959).

One point should be made clear; when Popper, Braithwaite and Reichenbach urge that there is no logical analysis appropriate to the actual thinking process in scientific discovery they are saying nothing which Peirce or Hanson would reject. Peirce’s intention is that the birth of new ideas can never satisfactorily be cleared up by psychological, sociological and historical investigations alone. One important task of a philosopher is to conduct a logical (conceptual) investigation of discovery. There can be good reasons, or bad, for suggesting one kind of hypothesis over another. The reasons may differ entirely from those that lead one to accept a hypothesis. Peirce wishes to show that reasoning *towards* a hypothesis is of a different kind than reasoning *from* a hypothesis. He realizes that the former “has usually been considered either as not reasoning at all, or as a species of Induction”[1] But he states: “I don’t think the adoption of a hypothesis on probation can properly be called induction; and yet it is reasoning” (8.388).

Many philosophers only concern themselves with analyzing the reasons for accepting a hypothesis. Hanson notes, “They begin with the hypothesis as given, as cooking recipes begin with the trout.” To study only the verification of a hypothesis leaves a vital question unanswered – namely, how hypotheses are “caught.” Natural scientists do not ‘start from’ hypotheses. They start from data. Peirce’s theory of abduction is concerned with the reasoning, which starts from data and moves towards hypothesis.

2.1.1 A general description of Peirce’s theory of abduction

Before analyzing Peirce’s theory of abduction it is important to clarify his classification of inference, which can be schematized as follows:

$$\text{Inference}\begin{cases}\text{Explicative (analytic or deductive)}\\[2em] \text{Ampliative (synthetic)}\quad\begin{cases}\text{Abductive}\\[2em] \text{Inductive}\end{cases}\end{cases}$$

[1] C. S. Peirce, *Letters to Lady Welby,* (Irwin Lieb, ed. [New York: Whitlock’s, 1953]) p.42

In explicative inference the conclusion necessarily follows from the premises, while in the ampliative inference the conclusion does not necessarily follow from the premises. The conclusion amplifies rather than explicates what is stated in the premises. All the empirical sciences use such reasoning. Moreover, it is the kind of reasoning that introduces new ideas into our store of knowledge.

Peirce's classification differs from the traditional classification because it includes a novel type of inference in addition to induction and deduction. Most logicians identify induction with synthetic reasoning. They fail to recognize the trichotomy of inferences because, Peirce thinks, they have a too "narrow and formalistic conception of inference"(8.228). These logicians generally confine their investigation of reasoning to its 'correctness,' "by which they mean, its leaving an absolute inability to doubt the truth of a conclusion so long as the premises are assumed to be true" (8.383). This amounts to confining their study to deduction.

Peirce insists that ampliative reasonings are twofold: Induction and Abduction. Abduction concerns itself with the reasons for adopting a hypothesis. The adoption of a hypothesis on probation cannot properly be called induction; and yet it is *reasoning* and though its *security* is low, its *uberty* is high" (8.388). Thus, from deduction to induction and to abduction the security decreases greatly, while the *uberty* increases greatly.

Broadly speaking, abduction covers, "all the operations by which theories and conceptions are engendered"(5.590). These operations are best manifested in the process of arriving at a scientific hypothesis. Peirce thinks this process is essentially inferential: "Although it is very little hampered by logical rules, nevertheless it is logical inference, asserting its conclusion only problematically or conjecturally, it is true, but nevertheless having a perfectly definite logical form" (5.188). Its form is:

The surprising fact is observed,
But if A were true, C would be a matter of course;
Hence, there is reason to suspect that A is true. (5.189)

Such a process is inferential because the hypothesis "is adopted for some reason, good or bad, and that reason, in being regarded as such, is regarded as lending the hypothesis some plausibility" (2.511n.).

Most writers who tackle Peirce's theory of abduction divide his thought roughly into two periods. The transition from one view to another was made around the turn of the century, but since this transition takes place over an extended period of time it is difficult to pinpoint a definite year. Peirce himself writes in 1910, "in almost everything I printed before the beginning of this century I more or less mixed up hypothesis [abduction] and induction"(8.227). Writers on Peirce vary greatly on this point. Judging from Peirce's writing, the best account is given by Burks; he names the year 1891, when Peirce had retired to his home near Milford, Pennsylvania, as the beginning of a transitional decade dividing the two periods.

In his earlier papers Peirce treats inference, and hence abduction, as an evidencing process. The three types of inferences are considered separate and independent forms of reasoning. Induction "infers the existence of phenomena such as we have observed in cases that are *similar*," while abduction "supposes something of a different kind from what we have directly observed, and frequently something which it would be impossible for us to observe directly"(2.640). For induction we generalize from a number of cases in which something is true, and by extension, infer that the same thing is probably true of a whole class. However, in abduction we pass from the observation of certain facts to the supposition of a general principle to account for the facts. Thus, induction can be said to be an inference from a sample to a whole, or from a particular to a general law; abduction is an inference from a body of data to an explaining hypothesis, or from effect to cause, "the former classifies, the latter explains"(2.636).

In papers written after 1891 Peirce widens the concept of inference to include methodological processes as well as evidencing processes. The three kinds of reasoning, while remaining distinguishable, become closely interlinked. Abduction furnishes the reasoner with the hypothesis, while induction is the method of testing and verifying (2.776). Peirce perceives the three kinds of reasoning as three stages of inquiry. Abduction invents or proposes a hypothesis; it is the initial proposal of a hypothesis on probation to account for the facts. Deduction explicates hypotheses, deducing from them the necessary consequences, which may be tested. Induction consists of the process of testing hypotheses. Thus, "Abduction is the process of forming an explanatory hypothesis. It is the only logical operation which introduces any new ideas; for induction does nothing but determine a value, and deduction merely evolves the necessary consequences of a pure hypothesis" (5.171).

The two periods of Peirce's thinking by no means exhibit two distinct theories of abduction. The second period certainly represents Peirce's mature judgment on the matter, but it is the logical consequence of the earlier theory and can only be understood clearly in light of the earlier theory.

2.1.2 The two periods of Peirce' theory of abduction

The earliest phase of Peirce's thought (from the earliest of his papers, until 1865) is very much a Kantian phase based on Kantian logic. One of the most important principles of Peirce's theory of knowledge, which he derived from Kant, is the doctrine that every cognition involves an inference. According to Kant there is no cognition until the manifold of sense has been reduced to unity. This reduction is accomplished by introducing a concept, which is not in and of itself a sensuous intuition. Thus, cognition requires some operation upon the manifold to bring it to unity, and Peirce writes in 1861, "An operation upon data resulting in cognition is an inference"(p. 21).

Peirce's conception of inference is shown more clearly in his theory of perceptual judgment: "Every judgment consists of referring a predicate to a subject. The predicate is thought, and the subject is only *thought-of*. The elements of the predicate are experiences or representations of experience. The subject is never experiential but only assumed. Every judgment, therefore, being a reference of the experienced or known to the assumed or unknown, is an explanation of a phenomenon by a hypothesis, and is in fact an inference"(p. 21).

Peirce regards all mental processes as inferential. Thus, "inference" includes not only deduction and induction but also *hypothesis* (what he will call later abduction), which is "an operation upon data resulting in cognition," or, "an explanation of a phenomenon by a hypothesis." However, in his early papers Peirce does not regard these as three distinct and irreducible forms of inference. His position is that all forms of inference may be reduced to Barbara. He writes in 1860, "It is clear that we draw no other inference from a thing's being a class other than what is directly expressed by Barbara namely *that whatever is true of an entire class is true of every member of the class;* hence all other syllogisms may be reduced to Barbara,"(Murphey, op. cit., pp. 21f)

Peirce's philosophy in the 1860's is based largely upon the notions of classical logic, and particularly upon the subject-predicate theory of the proposition. But the discovery of the logic of relations in the late 1860's introduces propositions which are not reducible to subject-predicate form. In 1870 Peirce published his first paper on the logic of relations and analyzed the syllogisms as a form of logical relations, rather than as fundamental formula of all argument (3.66). However, in "Deduction, Induction, Hypothesis" (2.619-644), the forms of induction and hypothesis are set up in a manner similar to that of 1868. Induction is the inference of the *rule* (major premise) and the *case* (minor premise) and *result* (conclusion), while hypothesis is the inference of a *case* from a *rule* and a *result*. The following example shows the relationships more clearly:

Deduction: Rule – All the beans from this bag are white
Case – These beans are from this bag
∴ Result – These beans are white

Induction: Case – These beans are from this bag
Result – These beans are white
∴ Rule – All the beans from this bag are white

Hypothesis: Rule – All the beans from this bag are white
Result – These beans are white
∴ Case – These beans are from this bag. (2.623)

"Induction is where we generalize from a number of cases of which something is true, and infer that the same thing is true of the whole class. As, where we find a certain thing to be true of a certain proportion of cases and infer that it is true of the same proportion of the whole class" (2.624). Hypothesis is where we find some surprising fact that would be explained by supposing that it is a case of a certain general rule, and thereupon adopt that supposition. This type of inference is called "making a hypothesis" (2.623). In this type of inference one should keep in mind that, "When we adopt a certain hypothesis, it is not alone because it will explain the observed facts, but also because the contrary hypothesis would probably lead to results contrary to those observed" (2.628). Peirce seems to hint here that hypothesis *selection* is involved in this kind of inference.

Explanatory hypotheses may be of widely different kinds and Peirce alludes to at least three: (1) the kind, which refers to facts, unobserved when hypotheses are made, but which are capable of being observed. For example, upon entering a room I find many bags containing different kinds of beans. On a table there is a heap of white beans; I may adopt the hypothesis that the heap is taken from a bag that contains white beans only. (2) There are hypotheses that are incapable of being observed. This is the case about historical facts: "Fossils are found, say, remains like those of fishes, but far in the interior of the country. To explain the phenomenon, we suppose the sea once washed over this land." And, "Numberless documents refer to a conqueror called Napoleon Bonaparte. Though we have not seen the man, yet we cannot explain what we have seen, namely, all these documents and monuments, without supposing that he really existed" (2.625). (3) Finally hypotheses refer to entities, which in the present state of knowledge are both factually and theoretically unobservable. For example, "The kinetic theory of gases is an illustration of this kind. These are the most important kinds of hypotheses in sciences" (Fann, 1970).

Thus, in the 1870's, abduction proper, the process of adopting a hypothesis, is barely touched upon here. This is due to the fact that Peirce regards 'inference' as essentially an evidencing process in this period.

According to the present theory induction and hypothesis are separate forms of inference, "The essence of an induction is that it infers from one set of fact to another set of similar facts, whereas hypothesis infers from facts of one kind to facts of another" (2.642). It is impossible to infer hypothetical conclusions inductively. However, it should be noted that even in this early formulation Peirce is not prepared to separate the two forms of inference absolutely: “When we stretch an induction quite beyond the limits of our observations, the inference partakes of the nature of hypothesis” (2.640). Induction and hypothesis, therefore, may be perceived as occupying opposite ends of the continuum of ampliative inference. In the later period Peirce stretches the concept of induction to include induction of characters, and abductions will appear to be a quite different kind of inference.

2.1.3 The decade between 1890 and 1900

During the years between 1890 and 1900 Peirce’s theory of abduction undergoes a fundamental change. Although the notion of abduction as the process of entertaining a hypothesis became quite explicit in the early 1890’s, the three kinds of reasoning were not as the three stages of inquiry until a decade later. This change does not constitute a sudden abandonment of one view in favor of another entirely different view, for the change was gradual and the roots of the latter view go further back.

In the years between 1891 and 1893 Peirce declares the following about the three forms of inference: “By hypothetic inference, I mean, as I have explained in other writings, an induction from qualities,” and, “By the hypothetic process, a number of reactions called for by one occasion get united in a general idea which is called out by the same occasion” (6.145). Induction is equated to the process of habit formation, while deduction is the process whereby the rule or habit is actualized in action. This is exemplified in the way a decapitated frog *reasons* when you pinch his hind legs: the habit serves as a major premise; the pinching is his minor premise, and the conclusion is the act of jumping away (6.144, 2.711, 6.286). Peirce’s concern here (in “The Law of Mind”) is merely to show that the three forms of inference have analogues in psychological phenomena. The treatment here seems little more than a restatement of the point already made in 1878 (2.643). Induction, hypothesis and analogy, “as far as their ampliative character goes, that is, as far as they conclude something not implied in the premises, depend upon one principle and involve the same procedure. All are essentially inferences from sampling” (6.40). This statement clearly belongs to his early thinking, for according to his later thinking only abductions involve additions to the fact observed. Induction can never originate any idea but simply confirms a hypothesis.

However, in Peirce's other writings of the early 1890's, characteristics of the later view of abduction are already explicit. Taking issue with the Positivists in his 1893 revision of the earlier article, "On the Natural Classification of Arguments," Peirce reiterates his contention that hypothetic inference is a legitimate and independent form of inference. A hypothesis "is adopted for some reason, good or bad, and that reason is being regarded as leading the hypothesis some plausibility" (2.511 n.1).

In a manuscript of notes from a projected but never completed, *History of Science*, written probably in the early 1890's[2] Peirce adopts a new term, "Retroduction," to designate what he used to call hypothesis, and mentions that this is the same as Aristotle's "abduction" (1.65).[3] Peirce contends, "Retroduction is the provisional adoption of a hypothesis, because every possible consequence of it is capable of experimental verification, so that the persevering application of the same method may be expected to reveal its disagreement with facts, if it does so disagree" (1.68). The conception of abduction is obviously stretched to include the methodological process as well as evidencing process. He begins to consider the reasons for adopting a hypothesis.

His conception of deduction and induction remains unchanged in the early 1890's. However, by 1898, deduction is clearly regarded as the process of tracing out the necessary and probable consequences of a hypothesis. Peirce writes, "Reasoning is of three kinds. The first is necessary, but it only professes to give us information concerning the matter of our own hypotheses…The second depends upon probabilities…The third kind of reasoning tries what *il lume naturale*…can do. It is really an appeal to instinct" (1.630). Induction is not yet treated as the process of testing a hypothesis. The basic idea is virtually expressed when Peirce, reflecting the "views of Whevell" in 1893, states that

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[2] It is dated in the *Collected Papers* as c.1896, but a more accurate date is provided by Wiener in *Studies in the Philosophy….* p. 344 n.5. He thinks it should be dated 1891-1892.

[3] In the later period Peirce commonly used the term *abduction* and sometimes retroduction or presumption. He seems to prefer *abduction* as the best designation. In support of his use of the term he refers to "the doubtful theory…that the meaning of the 25th chapter of the 2nd book of the *Prior Analytics* has been completely diverted from Aristotle's meaning by a single wrong word having been inserted by Appellicon where the original was illegible" (8.209). In 1901 he wrote a detailed investigation of this "doubtful theory" in "The Logic of Drawing History from Ancient Documents" (7.249). See also 2.776, 2.37n, 5.144.

"the progress in science depends upon the observation of the right facts by minds *furnished with appropriate ideas*" (6.604).

It is obvious that Peirce is on his way to regarding the three modes of inference as the three stages in scientific inquiry. An interesting point to be noted is the fact that up until the end of the 19th century, Peirce always listed the three modes of inference according to degrees of certainty, namely: deduction, induction, and hypothetic inference. After he comes to regard them as the three stages in an inquiry the list becomes: abduction, deduction, and induction.

2.1.4 Abduction, Deduction, and Induction as the three stages of Inquiry

The first full statement of Peirce's later theory of abduction is contained in his 1901 manuscript, "On the Logic of Drawing History from Ancient Documents." When surprising facts emerge, an explanation is required, and, "The explanation must be such a proposition as would lead to the prediction of the observed facts, either as necessary consequences or at least as very probable under the circumstances. A hypothesis then, has to be adopted which is likely in itself and renders the facts likely. This step of adopting a hypothesis as being suggested by the facts is what I call *abduction*" (7.202). A hypothesis adopted in this way could only be adopted on probation and must be tested. Peirce calls abduction the "First Stage of Inquiry" (6.469). "The first thing that will be done, as soon as a hypothesis has been adopted, will be to trace out its necessary and probable experimental consequences. This step is *deduction*" (7.203).

The next step is to test the hypothesis by conducting experiments and comparing the predictions drawn from the hypothesis with the actual results of the experiment. When we find that prediction after prediction is verified by experiment we begin to accord to the hypothesis a standing among scientific results. "This sort of inference it is, from experiments testing predictions based on a hypothesis, that is alone properly entitled to be called *induction*" (7.206).

The three kinds of inference now become three stages in a scientific inquiry. They are intimately connected as a *method*. Peirce's view on the relationship between the three modes of inference remains essentially the same from this date. He confines his attention mostly to scientific reasoning, and "inference" is mainly treated as a methodological process.

2.1.4.1 The main differences between abduction and Induction

According to the early view both abduction and induction are "synthetic" in the sense that something not implied in the premise is contained in the conclusion. The difference between the two lies in the results of the inferences. Induction is reasoning from particulars to a general law: abduction, from effects to cause. The former classifies while the latter explains. Under Peirce's present view *any* synthetic proposition, whether it is a non-observable entity or a generalization (so-called), insofar as it is for the first time entertained as possibly true, is a hypothesis arrived at by abduction. He states, "Any proposition added to observed facts, tending to make them applicable in any way to other circumstances than those under which they were observed may be called a hypothesis…By a *hypothesis*, I mean, not merely a supposition about an observed object…but also any other supposed truth from which would result such facts as have been observed, so when Van't Hoff, having remarked that the osmotic pressure of one-per-cent solutions of a number of chemical substances was inversely proportional to their atomic weights, thought that perhaps the same relation would be found to exist between the same properties of any other chemical substance" (6.524f.). Under the earlier view, this last example of abduction would have been called a "generalization" which would only be the result of induction. Under the present view such generalization is *suggested* by abduction and only *confirmed* by induction. In fact, Peirce now considers "laws" or "generalizations" explanatory hypotheses. He writes, "An explanation of a Phenomenon as the term is used in the so-called 'descriptive' sciences…consists in showing that the observed phenomenon follows logically, either necessarily or probably, from Explanatory Hypothesis required by sound logic. (Since science begins in observation, followed by explanation, which in time leads to classification of phenomena, and classification ultimately results in the discovery of law applicable to further explanation)."[4]

[4] *Logic Notebook, C. S. Peirce Papers,* Houghton Library, Harvard University, p. 294 (1908).

The relationship between abduction and induction is now very clear, "The induction adds nothing. At the very most it corrects the value of a ratio or slightly modifies a hypothesis in a way, which has already been contemplated as possible. Abduction, on the other hand, is merely preparatory. It is the first step of scientific reasoning, as induction is the concluding step…They are the opposite poles of reason, the one the most ineffective, the other the most effective of arguments. The method of either is the very reverse of the other's…Abduction seeks a theory. Induction seeks for facts" (7.217-218).

2.1.4.2 Abduction as hypothesis construction or Abduction as hypothesis selection?

Of what does abduction consist? Is it the logic of *constructing* a hypothesis, or the logic of *selecting* a hypothesis from among many possible ones? Peirce himself did not always keep this distinction in mind and often treated them as the same question. In some of his writings he maintains, "Abduction consists in studying facts and devising a theory to explain them" (5.145); "Abduction is the process of forming an explanatory hypothesis" (5.171); or abduction "consists in examining a mass of facts and in allowing these facts to suggest a theory" (8.209). In other writings he regards abduction as "the process of choosing a hypothesis" (7.219). To understand the nature of abduction it is necessary to investigate the relationship between hypothesis construction and selection.

As mentioned before, abduction is concerned with analyzing the *reasons* for proposing a hypothesis. The question arises: Is abduction concerned with the reasons for constructing a hypothesis in a certain way, or is it concerned with the reasons for preferring one hypothesis over many other possible ones? At the outset these seem to be two entirely different questions, but in practice the way one constructs a hypothesis is innately connected with the notion of choosing the best hypothesis. The purpose of constructing a hypothesis is to *explain* some facts. But from any given set of facts there may be a countless number of possible explanatory hypotheses. "Consider the multitude of theories that might have been suggested. A physicist comes across some new phenomenon in his laboratory. How does he know but the conjunctions of the planets have something to do with it or that it is not perhaps because the dowager empress of

China has at that same time a year ago chanced to pronounce some word of mystical power or some invisible jinnee may be present" (5.172); or "his daughter having on a blue dress, he having dreamed of a white horse the night before, the milkman having been late that morning, and so on?" (5.591): In one sense the proposing of a hypothesis is no problem at all. But of the trillions of hypotheses that might be made only one is true. The problem of constructing a good hypothesis is, thus, analogous to the problem of choosing a good hypothesis. The two questions, in practice, merge together.

This analysis is implicit in the following definition of abduction: "The first starting of a hypothesis and the entertaining of it, whether as a simple interrogation or with any degree of confidence, is an inferential step which I propose to call abduction. This will include a preference for any one hypothesis over others which would equally explain the facts as long as this preference is not based upon any previous knowledge bearing upon the truth of the hypotheses, nor on any testing of any of the hypotheses, after having admitted them on probation. I call all such inference by the peculiar name, *abduction…*" (6.525).

Peirce names three main considerations that should determine our choice of a hypothesis (7.220): In the first place, a hypothesis must be such that it will explain the surprising facts we have before us; in the second place, it must be capable of being subjected to experimental testing. This point is closely connected with the doctrine of Pragmatism; and, "In the third place, quite as necessary as consideration as either of those I have mentioned, in view of the fact that the true hypothesis is only one out of innumerable possible false ones, in view, too, of the enormous expensiveness of experimentation in money, time, energy, and thought, is the consideration of economy" (7.220).

Let us analyze the first consideration. The whole motive of our inquiry is to rationalize certain surprising facts by the adoption of an explanatory hypothesis. "The hypothesis cannot be admitted, even as a hypothesis, unless it be supposed that it would account for the facts or some of them. The form of inference, therefore, is this:

The surprising fact, C, is observed;
But if A were true, C would be a matter of course;
Hence, there is reason to suspect that A is true.

Thus, A cannot be abductively…conjectured until its entire content is already present in the premise, 'If A were true, C would be a matter of course'" (5.189). This explanation shows how the phenomenon would be produced, come about, or result in case the hypothesis were true. It may "consist in making the observed facts natural chance results, as the kinetical theory of gases explain facts; or it may render the fact necessary" (7.220).

2.1.5 Peirce's justification of abduction

A possible justification for abduction is that it is the only logical operation that introduces any new ideas. Deduction *explicates* and proves that something *must* be; induction *evaluates* and shows that something *actually* is operative. But abduction merely suggests that something *may be* (may be and may be not) (5.171, 6.475, 8.238).

There are two aspects to the problem of justification. The formal aspect is concerned with the *rationale* of abduction. The only justification for a hypothesis is that it *explains* the facts (1.89, 1.139, 1.170, 2.776, 6.606). Now to explain a fact is to show that it is a necessary or a probable result from another fact, known or supposed. Thus this part of the problem is simply a question of reducing any given abductive inference to a corresponding deduction. If the latter turns out to be valid, the correctness of the abduction is guaranteed.

The form of abduction:

The surprising fact, C, is observed;
But if A were true, C would be a matter of course;
Hence, there is reason to suspect that A is true (5.189).

This is valid because the corresponding deduction is valid:

If A were true, C would be a matter of course,

A is true;

Hence, C is true.

In answer to the question "how abduction is possible?" Peirce replies, "The validity of a presumptive adoption of a hypothesis being such that its consequences are capable of being tested by experimentation, and being such that the observed facts would follow from it as necessary conclusions, that hypothesis is selected according to a method which must ultimately lead to the discovery of the truth" (2.781). And, "Its only justification is that its method is the only way in which there can be any hope of attaining a rational explanation" (2.777, cf. 5.145, 5.171, 5.603). In other words, Peirce wants to say that the validity of abduction depends upon the validity of the whole scientific method.

The above "justification" seems to be merely a restatement of what abduction *is* instead of providing an independent "validity" for abduction. In fact, it is doubtful whether Peirce was ever satisfied with his justification of abduction. As late as 1910, he writes, "as for the validity of [abduction], there seems at first to be no room at all for the question of what supports it…But there is a decided leaning to the affirmative side and the frequency with which that turns out to be an actual fact is to me quite the most surprising of all wonders of the universe" (8.238).

Elsewhere Peirce tries to account for this "wonder" about the remarkable success which abduction has achieved in leading to true theories about nature. Peirce contends that the reasonable supposition is that man has come to the investigation of nature with a special aptitude for choosing correct theories. This facility is derived from his instinctive life through the process of evolution. Thus, the achievements of abduction are due to the fact that human intellect is peculiarly adapted to the comprehension of the laws of nature.

In their immediacy abductions are often merely guesses; it is quite possible for us to guess incorrectly on the first few occasions. But in the long run, "before very many hypotheses shall have been tried, intelligent guessing may be expected to lead us to the one which will support all tests, leaving the vast majority of hypotheses unexamined" (6.530). This is the foundation upon which abductive inference rests.

Peirce's treatment of the validity of abduction is one of the most unsatisfactory features of his theory. The claim that abduction is *necessarily valid in itself* is essential to the whole theory, but he seems unable to provide a clear-cut justification for such a claim. The affinity of mind with nature is a hypothesis, which can only be arrived at by abduction and thus must not be used to support the validity of abduction. This failure to provide an independent justification for abduction remains a difficult problem for contemporary philosophers who maintain that there is a logic to discovery.

2.2 Cifarelli and the role of Abduction

Cifarelli approaches abduction from a different point of view than Peirce; part of his research is concerned with the relationships between abductive approaches and problem-solving strategies. The purpose of his work is to clarify the processes by which learners construct new knowledge in mathematical problem solving situations, with particular focus on instances where the learner's emerging abductions or hypotheses help to facilitate novel solution activity (Cifarelli, 1999). The basic idea is that an abductive inference may serve to organize, re-organize, and transform a problem solver's actions.

Cifarelli reveals that few studies of mathematical problem solving have specified precisely the role of abductive actions in the novel solution activity of solvers, but the research on problem posing (Silver, 1994; Brown and Walter, 1990) suggests ways that hypotheses play a prominent role in solvers' novel solution activity. According to Brown and Walter (1990), problem posing and problem solving are naturally related in the sense that new questions emerge as one is problem solving, that "we need not wait until *after* we have solved a problem to generate new questions; rather, we are logically obligated to generate a new question or pose a new problem in order to solve a problem *in the first place*" (p.114). Furthermore, Silver (1994) asserts that this kind of problem posing, "problem formulation or re-formulation, occurs within the process of problem solving" (p.19). Finally, the cognitive activity of "within-solution posing, in which one reformulates a problem as it is being solved" (Silver and Cai, 1996, p.523), may aid the solver to consider "hypothesis-based" questions and situations (Silver and Cai, 1996, p.529). According to Cifarelli, this illustrates both the dynamic, yet tentative nature of solvers' solution activity as well as the propensity of the solvers to abduce novel ideas about problems while in the process of solving them.

To this extent Cifarelli has conducted a study with the purpose of analyzing the problem posing and solving processes of the learners in mathematical problem solving situations, with particular focus on ways that the learner's emerging abductions or hypotheses help to facilitate their novel solution activity; "Their interpretations of a particular task may suggest to them additional questions and uncertainties, the consideration of which helps them construct goals for purposeful action…In this way, problem solving can be viewed as a form of abductive reasoning through which solvers mentally reflect upon and contemplate viable strategies to relieve cognitive tension, involving no less than their ability to form conceptions of, transform, and elaborate the problematic situations they face" (Cifarelli, 1999)

The following example given by Cifarelli (in "*Abduction, Generalization, and Abstraction in Mathematical Problem Solving*", 1998) may highlight the core of his work:

> Marie is a student who was given a set of algebra word problems, designed by Yackel (1984) to induce problematic situations.
> Marie had to solve the first problem involving the depths of two lakes, and then she was asked to solve eight follow-up tasks, each a variation of the original problem. The problems were designed in such a way to have a range of similar problem solving situations and hence develop ideas about "problem sameness" in the course of her on-going activity. The third problem had insufficient information; initially, Marie guided by the sameness of the problem tried to solve it in the same way she had solved the previous two; very soon she realized that it was not possible and that became for her a novel situation. The abduction took place at this point, namely Marie needed to find an explanation of her failure.
> <<…The same way (she smiles, then displays a facial expression suggesting sudden puzzlement) impossible!! It strikes me suddenly that there might not be enough information to solve this problem (she re-reads and reflects on her work) I suspect I'm going to need to know the height of one of these things (solver points to both containers in her diagram). I don't know though, so I am going to go over here all the way through>>

Cifarelli’s analysis of Marie’s process is as follows:

<<Marie’s anticipation that “the same way” would not work was followed by her *abduction* that the problem did not contain enough information, later refined to the hypothesis that she needed more information about the relative heights of the unknowns. While the hypothesis contained elements of uncertainty, it helped organize and structure her subsequent solution activity, whereupon she explored and tested its plausibility as an explanatory device>> (p.7).

Cifarelli’s attention is focused on the abductive inference as a tool to enhance the search for further strategies when the application of a previous solution did not work. The hypothesis of the absence of enough information leads Marie to go through the problem again to verify the plausibility of her hypothesis, and then to construct the necessary data to solve the problem. Therefore, the researcher is not interested in the “typological aspect” of abduction, but in the role such a process plays on the problem-solving activities.

Another example of Cifarelli’s work is represented by the analysis of some episodes from interviews conducted with Jessica, one of the five graduate students in Mathematics Education who participated to the project. The five students were enrolled in a class, taught by the researcher, “The Use of Technology to Teach Middle and Secondary Mathematics.” The students were interviewed on three occasions throughout the course. These interviews took the form of problem solving sessions, where students solved a variety of algebraic and non-algebraic word problems while “thinking aloud.”

The following is an excerpt of an interview with Jessica and the researcher’s analysis of her work:

> Problem: Sally, an avid canoeist, decided one day to paddle upstream 6 miles. In 1 hour, she could travel 2 miles upstream, using her strongest stroke. After such strenuous activity, she needed to rest for 1 hour, during which time the canoe floated downstream 1 mile. In this manner of paddling for 1 hour and resting for 1 hour, she traveled 6 miles upstream. How long did it take her to make this trip?

Jessica, after having read the problem, commented she had seen a similar problem before but had not solved it.

> **Jessica:** *I have had one like this…and I'm not sure. I had a similar one in Dr. L's class. Upstream-downstream, airplane flying with the wind behind them. Professor L gave us a list of 100 problems. I looked them over and did not choose this one. I didn't do it, but I did watch other students do it. So I have not technically done this problem.* ***(Appears confident she can do it[5]).*** ***(Re-reads the problem; several seconds of reflection)***
> **Jessica:** *Okay, distance is 6 miles. Let's see…total time is 2 hours…we have to modify this because upstream means you are getting help and downstream means you are not…Oh, wait…****(reflection)*** *1 hour she travels 2 miles up…and she rests 1 hour…so it is not total is 2 hours. I read the last sentence…and I totally forgot what I was supposed to find...the total time. Okay…distance equals rate x time, so 1 hour, okay the distance is 2 miles, time is 1, and rate…****(long reflection)****…resting distance is –1, equals rate…1 um…so* ***(reflection; appears frustrated)****…I know I have to set an equation then…I could…****(reflection; facial expressions suggest she is puzzled)***

Cifarelli's comment is: *Jessica's comments indicate that even though she had seen others solve the problem before, she still had some difficulty solving the same problem. She continued to reflect upon the situation and then had an idea to do something different to solve the problem:*

> **Jessica: (long reflection, makes motions with her hands)** *okay! So she paddles first, then she rests. She goes +2, then –1, she goes +2, -1, she goes +2 and 1, 3, and she goes +2 again. So that's 1,2…9 hours she makes the trip. That's not how they did it in class.*

The interviewer questioned Jessica about her reasoning:

> **Interviewer:** *Ah, so you were thinking back to how they did it?*
> **Jessica:** *Well this reminded me of that problem. I was trying to do what they did. But when I tried to do it their*

[5] Comments in boldface describe the non-verbal actions of the solvers as inferred by the researcher.

> *way, and try to get some equation going, it didn't work. I had to try something else. So just apply logic to it, it's +2, -1, +2, -1, then set up an equation (sic) to see if it works.*

According to the author, Jessica's explanation indicates both the provisional aspect of her reasoning as well as the belief on her part that her ideas still need to be verified "to see if they work"; namely, she abduces an idea of what the problem might be about and then initiates the appropriate solution activity to test her abductive hypothesis.

Cifarelli's work goes on, asking Jessica to solve an extension of the aforementioned problem:

> ***Suppose after 4 hours on the river, Sally took a lunch break for 1 hour, during which time she floated downstream. How long did it take her to go the 6 miles up the river?***
>
> **Researcher's comment:** Jessica solved the follow-up task routinely. However, her solution surprised her and she demonstrated abductive reasoning in "making sense" of her solution.
>
> **Jessica:** *Okay…so paddling is +2, resting is –1, so she rests another hour for lunch that's another –1 so first hour is +2:-1, +2:-1, and she did lunch, so that's another –1. So, 1 there…she rests an hour, and another hour, so those 2 cancel out going back to 1 hour. So we have +2:-1,+2:-1,…1,2,…6…11,12 hours to make trip with lunch break.*
>
> **Jessica:** *What!?* ***(She appears surprised by her result; long period of reflection).*** *Yeah, I guess that 1-hour sets you back.* ***(Several seconds of reflection)***
>
> **Interviewer:** *What are you thinking?*
>
> **Jessica:** *well, I was going to say that it would have been 10 hours, but I guess…maybe you have to add a whole 'nother cycle?* ***(Reflection)*** *Let's see.* ***(She annotates her diagram)****. Yeah, you add +2:-1 to make up for that resting time, that one –1, to put an extra 2 plus –1 in there, cause that just cancels that whole one out there, and gives 3 more than 9 total. So I guess it is 12, yeah…Sally's crazy! 12 hours.*

Cifarelli's comments about Jessica's solution underline the fact that in solving the initial task, her abduction helped her make sense of her realization that the way she has seen others solve a similar problem would not work, and upon solving the extension of the canoe problem, her abduction helps her make sense of the surprising fact that inserting a one-hour rest time into the previous task changed the solution by 3 hours (and not a mere 1 hour like she initially expected).

Finally, Cifarelli's work seems to uncover a form of novel problem posing that has not been addressed in the problem solving literature (like English, 1997: Silver, 1994; Silver and Cai, 1996).

2.3 Magnani and Abduction

> More than a hundred years ago, the great American philosopher Charles Sanders Peirce coined the term "abduction" to refer to inference that involves the generation and evaluation of explanatory hypotheses. The study of abductive inference was slow to develop, as logicians concentrated on deductive logic and on inductive logic based on formal calculi such as probability theory. In recent decades, however, there has been renewed interest in abductive inference from two primary sources. Philosophers of science have recognized the importance of abduction in the discovery and evaluation theories, and researchers in artificial intelligence have realized that abduction is a key part of medical diagnosis and other tasks that require finding explanations. Psychologists have been slow to adopt the terms "abduction" and "abductive inference" but have been showing increasing interest in causal and explanatory reasoning.
> This abduction is now a key topic in cognitive science, the interdisciplinary study of mind and intelligence. Lorenzo Magnani's new book contributes to this research in several valuable ways. First, it nicely ties together the concerns of philosophers of science and AI researchers, showing, for example, the connections between scientific thinking and medical expert systems. Second, it lays out a useful general framework for discussion of various kinds of abduction. Third, it develops important ideas about aspects of abductive reasoning that have been relatively neglected in cognitive science, including the visual and temporal representations and the role of abduction in the withdrawal of hypotheses. The author has provided a fine contribution to the renaissance of research on explanatory reasoning. (Paul Thagard)

The book starts with the words written by Paul Thagard to underline the important contribution offered by Lorenzo Magnani to the wide scenario of abduction. His attempt is to explore abduction, meant as inference to explanatory hypotheses, and the aim is to integrate philosophical, cognitive, and computational issues, while also discussing some cases of reasoning in science and medicine, in order to illustrate the problem-solving process and to propose a unified epistemological model of scientific discovery, diagnostic reasoning, and other kinds of creative reasoning.

The study of diagnostic, visual, spatial, analogical and temporal reasoning has demonstrated that there are many ways of performing intelligent and creative reasoning that cannot be described with only the help of classical logic. However, non-standard logic has shown how we can provide rigorous formal models of many kinds of abductive reasoning such as the ones involved in defeasible and uncertain inferences. To this extent Magnani starts introducing two kinds of abduction, *theoretical* and *manipulative*, in order to provide an integrated framework to explain some of the main aspects of both creative and *model-based reasoning* effects engendered by the practice of science.

2.3.1 Theoretical Abduction

Theoretical abduction is the process of *inferring* certain facts and/or laws and hypotheses that render some sentences plausible, that *explain* or *discover* some (eventually new) phenomenon or observation; it is the process of reasoning in which explanatory hypotheses are formed and evaluated. For instance, if we see a broken horizontal glass on the floor[6] we might explain this fact by postulating the effect of wind shortly before: this is not certainly a deductive consequence of the glass being broken (a cat may well have been responsible for it).

There are two main epistemological meanings of the word abduction (Magnani, 1988, 1991): (1) abduction that only generates "plausible" hypotheses (*creative* or *selective*) and (2) abduction considered as *inference to the best explanation*, which also evaluates hypotheses. *Creative* abduction deals with the whole field of the growth of scientific knowledge (Blois, 1984).

S*elective* abduction tends to produce hypotheses for further examination that have some chance of turning out to be the best explanation.

[6] This event constitutes in its turn an *anomaly* that needs to be solved/explained.

We can consider the following example, dealing with diagnostic reasoning and illustrated in syllogistic terms (see also Lycan, 1988):

1. If a patient is affected by a pneumonia, his/her level of white blood cells increases
2. John is affected by pneumonia
3. John's level of white blood cells increases

(This syllogism is known as Barbara)

By deduction we can infer (3) from (1) and (2). Two other syllogisms can be obtained from Barbara if we exchange the conclusion (or Result, in Peircean terms) with either the major premise (the Rule) or the minor premise (the Case): by induction we can go from a finite set of facts, like (2) and (3), to a universally quantified generalization - also called categorical inductive generalization. Like the piece of hematologic knowledge represented by (1). Starting from knowing – selecting – (1) and "observing" (3) we can infer (2) by performing a selective abduction.

Thus, *selective* abduction is the making of a preliminary guess that introduces a set of plausible diagnostic hypotheses, followed by deduction to explore their consequences, and by induction to test them with available patient data; (1) to increase the likelihood of a hypothesis by noting evidence explained by that one, rather than by competing hypotheses, or (2) to refute all but one.

Inside the *theoretical* abduction, Magnani defines the Visual abduction, bearing in mind Peirce's assertion, namely, that all thinking is in signs, and signs can be icons, indices, or symbols. Moreover, all inference is a form of sign activity, where the word sign includes "feeling, image, conception, and other representation" (CP. 5.283), and in Kantian language, all synthetic forms of cognition. Therefore, *Visual abduction*, a special form of non-verbal abduction, occurs when hypotheses are instantly derived from a stored series of previously similar experiences. It covers a mental procedure that tapers into a non-inferential one, and falls into the category called "perception." Philosophically, *perception* is viewed by Peirce as a fast and uncontrolled knowledge-production procedure. Perception is in fact a vehicle for the instantaneous retrieval of knowledge that has been previously structured in our minds through inferential processes.

Peirce remarks: “Abductive inference shades into perceptual judgment without any sharp line of demarcation between them” (Peirce, 1955c, p.304): By perception, knowledge constructions are so instantly reorganized that they become habitual and diffuse and do not need any further testing: “[…] a fully accepted, simple, and interesting inference tends to obliterate all recognition of the uninteresting and complex premises from which it was derived” (CP 7.37). Many visual stimuli – that can be considered the “premises” of the involved abduction – are ambiguous, yet people are adept at imposing order on them: “We readily form such hypotheses as that an obscurely seen face belongs to a friend of ours, because we can thereby explain what has been observed” (Thagard, 1988, p. 53). This kind of image-based hypothesis formation can be considered as a form of *visual* (or iconic) *abduction*.

Peirce gives another interesting example of model-based abduction related to sense activity: “A man can distinguish different textures of cloth by feeling: but not immediately, for he requires to move fingers over the cloth, which shows that he is obliged to compare sensations of one instant with those of another”(CP, 5.221). This surely suggests that abductive movements also have interesting extra-theoretical characters and that there is a role in abductive reasoning for various kinds of manipulations of external objects.

In conclusion, for Peirce all knowing is *inferring* and inferring is not instantaneous, it happens in a process that needs an activity of comparisons involving many kinds of models over a more-or-less considerable lapse of time. This is not in contradiction with the fact that for Peirce the inferential and abductive character of creativity is based on the instinct (the mind is “in tune with the nature”) but does not have anything to do with irrationality and blind guessing.

Human beings and animals have evolved in such a way that now they are able to recognize habitual and recurrent events and to “emotionally” deal with them, like in cases of fear, that appears to be a quick explanation that some events are dangerous. During the evolution such abductive types of recognition and explanation settled in their nervous systems: we can abduce “fear” as a reaction to a possible external danger, but also when confronting a different type of evidence, like in the case of “reading a thriller”(Oatley, 1996).

In all these examples Peirce is referring to a kind of hypothetical activity that is inferential but not verbal, where “models” of feeling, seeing, hearing, etc., are efficacious when used to build both habitual abductions of everyday reasoning and creative abductions of intellectual and scientific life. We have to remember that visual and analogical reasoning is productive in scientific concept formation too; scientific concepts do not pop out of our heads, but are elaborated in a problem-solving process that involves the application of various procedures: this process is a *reasoned process*.

As we have seen, the general objective is to consider how the use of visual mental imagery in thinking may be relevant to hypothesis generation and scientific discovery. In this research area the term “image” refers to an internal representation used by humans to retrieve information from memory. Many psychological and physiological studies have been carried out to describe the multiple functions of mental imagery processes: there exists a visual memory (Paivio, 1975) that is superior in recall; humans typically use mental imagery for spatial reasoning (Farah, 1988); images can be rebuilt in creative ways (Finke, and Slayton, 1988); they preserve the spatial relationships, relative sizes, and relative distances of real physical objects (Kosslyn, 1980); for a more complete list, see Tye (1991).

Kosslyn introduces visual cognition as follows:

> Many people report that they often think by visualizing objects and events […] we will explore the nature of visual cognition, which is the use of visual mental imagery in thinking. Visual mental imagery is accompanied by the experience of seeing, even though the object or event is not actually being viewed. To get an idea of what we mean by visual mental imagery, try to answer the following questions: […] How many windows are there in your living room? If an uppercase version of the letter n were rotated 90^0 clockwise, would it be another letter? (Kosslyn and Koenig, 1992, p.128)

We can build visual images on the basis of visual memories but we can also use the recalled visual image to form a new image, one we have never actually seen. Certainly, imagery is used in everyday life, as illustrated by the previous simple answers, nevertheless imagery has to be considered as a major medium of thought, as a mechanism for thinking relevant to hypothesis generation. Some hypotheses naturally take a pictorial form: the hypothesis that the earth has a molten core might be better represented by a picture that shows solid material surrounding the core.

There has been little research on the possibility of visual imagery representations of hypotheses, despite abundant reports (e.g., Einstein and Faraday) that imaging is crucial to scientific discovery, but also in creative literary and artistic realizations (Koestler, 1964; Shepard, 1988, 1990). Einstein described having imaged the consequences of traveling at the speed of light, which led him to the discovery of the theory of special relativity. Faraday claimed to have visualized lines of force that emanated from electrical and magnetic sources, leading to the modern conception of electromagnetic fields.

Moreover, it is well-known that the German chemist Kekulé, used spontaneous imagery to discover the structure of benzene; Watson and Crick have reported the use of mental imagery in the interpretation of diffraction data and in the determination of the structure of the DNA molecule (Holton, 1972; Miller, 1984, 1989; Magnani, Civita, and Previde Massara, 1994; Nersessian, 1995a; Shepard, 1988, 1990; Thagard, Gochfeld, and Hardy, 1992; Tweney, 1989).

Thus, after illustrating the computational imagery representation scheme proposed by Glasgow and Papadias (1992), together with certain cognitive results, Magnani will explore whether a kind of hybrid imagery/linguistic representation architecture can be improved and used to model image-based hypothesis generation; i.e. to delineate the first cognitive and computational features of what he call *visual abduction*.

The central theme of the recent imagery debate in cognitive science concerns the problem of representation. How can we represent images? Are mental images represented depictively in a picture, or like sentences of descriptions in a syntactic language?
According to Kosslyn's *depictionist* or *pictorialist* view (Kosslyn, 1983), mental images are quasi-pictures represented in a specific medium called the visual buffer in the mind. Kosslyn's model of mental imagery proposes three classes of processes that manage images in the visual buffer: the *generation process* forms an image exploiting visual information stored in long-term memory, the *transformation process* (for example,

rotation, translation, reduction in size, etc.,) modifies the depictive image or views it from different perspectives, and the *inspection process* explores patterns of cells to retrieve information such as shape and spatial configuration. According to Pylyshyn's *descriptionist* view (1981, 1984) mental imagery can be explained by the tacit knowledge used by humans when they simulate events rather than by a pictorialist view related to the presence of a distinctive mental image processor.

According to Kosslyn's cognitive model, the knowledge representation scheme of mental imagery is composed of two different levels of reasoning, *visual* and *spatial*, the former concerned with what an image looks like, and the latter depending on where an object is located relative to other objects. The different representations of these methods of reasoning exist at the level of working-memory and are generated from a *descriptive representation* of an image stored in long-term memory in a hierarchical organization. Information is accessed from long-term memory by means of standard retrieval, procedural attachment and inheritance techniques.

According to Magnani, we can consider spatial representations as descriptive. Thus, they are expressed by propositions containing predicates such as spatial relationships and arguments as imaginable objects.

The spatial representation does not add information that cannot be expressed by propositions. Notwithstanding this, the spatial representation is not computationally equivalent to a descriptive one. In several imagery-related tasks (e.g. inspecting) spatial representation may reduce the computational complexity of the solution: the symbolic array adds more constraints to the search. As the spatial representations are depictive, and denote the important spatial relations among parts of the image, they are useful in the development of problem-solving devices related to the inspection and transformation of images.

The use of imagery in scientific discovery illustrates a mechanism of thinking relevant to hypothesis generation. Imagery also involves the simulation of image transformations in order to anticipate the consequences of an action or event; constructing novel images through operations such as compose, superimpose, and put, allows us to detect information not previously observed.

Having illustrated many issues concerning the phenomenon of imagery, which is commonly and consciously experienced as the ability to form, transform and inspect an image-like representation of a scene, and having indicated that such representations play a role in problem-solving strategies involving visual or spatial properties of an image, Magnani considers, from a computational philosophy perspective, a visual abductive problem-solving strategy.

Although there is considerable agreement concerning the existence of a high-level visual and spatial medium of thought as a mechanism relevant to abductive (selective and creative) hypothesis generation, the underlying cognitive processes involved are still not well understood. Notwithstanding this, Magnani will attempt to work around this gap in our understanding: although describing a model able to “imitate” the real ways the human brain works when it makes visual abductions would be best, his primary concern is its expressiveness and inferential adequacy, rather than its explanatory and predictive power as regards psychological research.

According to Magnani we can face an *initial* (eventually) observed image in which we recognize a problem to solve. For example, given a visual or imagery datum, we may have: (1) to explain the absence of an object; (2) explain why an object is in a particular position; (3) explain how an object can achieve a given task moving itself and/or interacting with the remaining objects in the scene/image; (4) to show how we can recognize an object as having significance (for instance the recognition of a stone as a toll) (Shelley, 1996).

How can “visual” reasoning perform these explanations? To answer this question it is necessary to show how visual abduction may be relevant to hypothesis generation, that is, how an *image-based explanation* is able to solve the problem given in the initial image.

Faced with the initial image, in which we have previously recognized a problem to solve, as stated above, we have to work out an *imagery hypothesis* that can explain the problem-data.[7] Thus, the formed image acquires a hypothetical status in the inferential abductive process at hand.

1) We have to *select* from long-term memory a visual (imagery) description that is able to explain the anomaly that needs to be solved; 2) we have to justify the presence/absence of a given object in a scene selecting a suitable imagery explanatory hypothesis; 3) for instance we have to visually solve the well-known monkey-banana problem: every formed visual representation of the effect of a sequence of actions the monkey can perform may be considered as a hypothesis generation. Such a hypothesis, if successful, is viewed as the one selected that gives a solution to the problem; 4) a slightly differently selected version of the initial image can perform the task of giving sense to an object.

The generation of a "new" imagery hypothesis can be considered the result of the *creative* abductive inference previously described; in this respect we can consider how the imagery representations of new hypotheses lead to scientific discovery. The selection of an imagery hypothesis from a set of pre-enumerated imagery hypotheses, stored in long-term memory, also involves abductive steps, but its creativity is much weaker: this type of visual abduction can be called *selective*.

All we can expect of visual abduction is that it tends to produce imagery hypotheses that have some chances of turning out to be the best explanation. Visual abduction will always produce hypotheses that give at least a partial explanation, and therefore have a small amount of initial plausibility. In this respect abduction is more effective than the blind generation of hypotheses.

2.3.2 Manipulative Abduction

Manipulative abduction happens when we are thinking *through* doing and not only, in a pragmatic sense, about doing. So the idea of manipulative abduction goes beyond the well-known role of experiments as capable of forming new scientific laws by means of the results (nature's answers to the investigator's question) they present, or of

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[7] When discussing some problems related to the abductive reasoning, Bayesian networks, perception, and vision, also Poole (2000) underlines that in vision we can think of a scene *causing* the image: "the scene produces the image, but the problem of vision is, given an image, to determine what is in the scene", that it is an abductive task.

merely playing a predictive role (in confirmation and in falsification). Manipulative abduction refers to an extra-theoretical behavior that aims at creating communicable accounts of new experiences to integrate them into previously existing systems of experimental and linguistic (theoretical) practices.

The existence of this kind of extra-theoretical cognitive behavior is also testified by the many everyday situations in which humans are perfectly capable of performing very efficacious (and habitual) tasks without the immediate possibility of realizing their conceptual explanation. In some cases the conceptual account for doing these things is at one point present in the memory, but has now deteriorated, and it is necessary to reproduce it. In other cases the account has to be constructed for the first time, like in creative settings of manipulative abduction in science. Hutchins (1995) illustrates the case of a navigation instructor that for 3 years performed an automatized task involving a complicated set of plotting manipulations and procedures.

The insight concerning the conceptual relationships between relative and geographic motion came to him suddenly, “as lay in his bunk one night.” This example explains that many forms of learning can be represented as the result of the capability of giving conceptual and theoretical details to already automatized manipulative executions. The instructor does not discover anything new from the point of view of the objective knowledge about the involved skill; however, we can say that his conceptual awareness is new from the local perspective of his individuality.

In this kind of action-based abduction the suggested hypotheses are inherently ambiguous until articulated into configurations of real or imagined entities (images, models or concrete apparatuses and instruments). In these cases only by experimenting we can discriminate between possibilities: they are articulated behaviorally and concretely by manipulations and then, increasingly, by words and pictures.

Some common features of these tacit templates that enable us to manipulate things and experiments in science are related to 1) Sensibility to the aspects of the phenomenon, which can be regarded as *curious* or *anomalous*; 2) Preliminary sensibility to the *dynamical* character of the phenomenon, and not to entities and their properties, one common aim of manipulations is to practically reorder the dynamic sequence of events in a static spatial sequence that should promote a subsequent bird’s-eye-view (narrative or visual-diagrammatic); 3) Referral to experimental manipulations that exploit *artificial apparatus* to free new possibly stable and repeatable sources of information about hidden knowledge and constraints; 4) Various contingent ways of epistemic acting: *looking* from different perspectives, *checking* the different information available; *comparing* subsequent events, *choosing*, *discarding*, *imaging* further manipulations, *re-ordering* and *changing* relationships in the world by implicitly *evaluating* the usefulness of a new order.

Manipulative abduction represents a kind of redistribution of the epistemic and cognitive effort to manage objects and information that cannot be immediately represented or found internally.

The interplay between manipulative and theoretical abduction consists of a superimposition of internal and external, where the elements of the external structures gain new meanings and relationships to one another, thanks to the constructive explanatory theoretical activity (for instance Faraday’s new meanings in terms of curves and lines of force).

In this light, Powers (1973) studied behavior, considering it as a *control of perception* and not only as controlled by perception. Flach and Warren (1995) used the term “active psychophysics” to illustrate that “the link between perception and action […] must be viewed as a dynamic coupling in which stimulation will be determined as a result of subject actions. It is not simply a two-way street, but a circle” (p.202). Kirsh (1995) describes situations (e.g., grocery bagging, salad preparation) in which people use action to simplify choice, perception, and reduce demands for internal computation through the exploitation of spatial structuring.

We know that theoretical abduction certainly illustrates much of what is important in abductive reasoning, especially the objective of selecting and creating a set of hypotheses (diagnoses, causes, hypotheses) that are able to dispense good (preferred) explanations of data (observations), but fail to account for many cases of explanations occurring in science or in everyday reasoning when the exploitation of the environment is crucial. The concept of manipulative abduction is devoted to capturing the role of action in many interesting situations: action provides otherwise unavailable information that enables the agent to solve problems by starting and performing a suitable abductive process of generation or selection of hypotheses.

From the point of view of everyday situations manipulative abductive reasoning exhibits very interesting features: 1) action elaborates a *simplification* of the reasoning task and a redistribution of effort across time (Hutchins, 1995), when we "need to manipulate concrete things in order to understand structures which are otherwise too abstract" (Piaget, 1974), or when we are in the presence of *redundant* and unmanageable information"; 2) action can be useful in the presence of *incomplete* or *inconsistent* information – not only from the "perceptual" point of view – or of a diminished capacity to act upon the world: it is used to get more data to restore coherence and to improve deficient knowledge; 3) action as a *control of sense data* illustrates how we can change the position our body (and/or of the external objects) and how to exploit various kinds of prostheses (Galileo's telescope, technological instruments and interfaces) to get various new kinds of stimulation: action provides some tactile and visual information (e.g., in surgery), otherwise unavailable; 4) action enables us to build *external artifactual models* of task mechanisms instead of the corresponding internal ones, that are adequate to adapt the environment to an agent's needs.

Artificial Intelligence research has developed many computational tools for describing the representation and processing of information. Cognitive psychologists have found these tools valuable for developing theories about human thinking and for their experimental research.

To escape relativism, epistemology is usually considered as the normative theory of objective knowledge, and thus does not need to take into account what psychology determines as the nature of individuals' belief systems. Logic and epistemology are concerned with how people ought to reason, whereas psychology is supposed to describe how people actually think.

Empirical studies of cognitive psychology are descriptive: they are dedicated to the investigation of mental processes and are concerned with normative issues only in order to characterize people's behavior relative to assumed norms. AI, when examined as cognitive modeling, is normally descriptive: only when it is concerned with improving on people's performances does it become involved with what is normative.

Epistemology, AI and cognitive psychology can be used together to develop models that explain how humans think (Thagard, 1988, 1996).

If abduction is considered as inference to the best explanation, abduction is epistemologically classified not only as a mechanism for selection (or for discovery), but also for justification.

2.4 Abduction and Mathematics

Mathematicians and mathematics educators have recognized the influence of abductive processes in mathematical thinking, although under different names. Lakatos (1976) acknowledged the nonlinearity of inferential reasoning, stating that, "discovery does not go up or down, but it follows a zigzag path; prodded by counterexamples, it moves from the naïve conjecture to the premise and then turns back again to delete the naïve conjecture and replace it with a theorem."

Mason (1995) points out that in trying to avoid difficulties, "the curriculum turns everything into behavior, avoids awareness, assumes deduction, tolerates induction, and ignores abduction."

Accounts of mathematics learning have long acknowledged the importance of autonomous cognitive activity, with particular emphasis on the learners' abilities to initiate and sustain productive patterns of reasoning in problem-solving situations. Nevertheless, most accounts of problem-solving performance have been explained in terms of inductive and deductive reasoning, containing little explanation of the novel actions solvers often perform prior to introducing formal algorithmic procedures into their actions. For example, cognitive models of problem solving seldom address the solver's idiosyncratic activity, such as the generation of novel hypotheses, intuitions, and conjectures, even though these processes are seen as crucial tools through which mathematicians ply their craft (Anderson, 1995; Burton, 1984; Mason, 1995).

2.5 Proofs and Proving

Much has been said and is still being said about proof, and amongst the questions that have arisen is: *What is a mathematical proof? What does prove mean in mathematics? How do we teach a mathematical proof? What is the role of proof in mathematics?*

Throughout the 20th century mathematicians and mathematics educators shared many differing positions about this issue. The tenet of proof has been analyzed from different points of view (pedagogical, historical, and cognitive), for this reason we must differentiate between proof as *product*, proof as *process* and the *teaching of proof*.

2.5.1 Proof as product

The first half of the 20th century was characterized by the search for precision and rigor (even though formalism has very ancient descendants**:** from the Greeks with Aristotle and Plato to Leibniz (1646-1716) and Frege (1848-1925) to arrive at Hilbert (1862-1943)). One very famous instance was a group of French mathematicians who wrote under the name of Bourbaki:

> The mathematics method teaches one…to find the common ideas buried under the external apparatus of detail appropriate to each of the theories considered, to single out these ideas and to exhibit them.
>
> (Bourbaki, 1971, p.26)

Such an approach led to the 60's where a great emphasis was given to formal proof, considered the most important characteristic of modern mathematics, indeed impressive work was done during the first 50 years of the century in clarifying the very foundations of mathematics, work that demonstrated the enormous power of formal systems constructed step by step from a base of definitions, axioms, and rules of inference.

Among the results of such a work we can find the birth of three main schools of thought going under the names of *Logicism, Formalism, and Intuitionism.*

Gottlob Frege, the German philosopher, logician and mathematician (1848-1925) began the school of *Logicism* in about 1884. Bertrand Russell rediscovered the school about eighteen years later. Other early logicists were Peano and Russell's co-author of Principia Mathematica, A. N. Whitehead. The purpose of Logicism is to show that classical mathematics is part of logic, and to this extent Russell and Whitehead created the *Principia Mathematica,* which was published in 1910. The *Principia* may be considered as a formal set theory: although the formalization was not entirely complete, Russell and Whitehead thought that it was and planned to use it to show that mathematics can be reduced to logic. They showed that all classical mathematics, known in their time, could be derived from set theory and hence from the axioms of the *Principia Mathematica*. Consequently, what remained to be done was to show that all the axioms of *Principia Mathematica* belong to logic. Snapper (1979) states that in order to understand Logicism, it is important to clearly understand what the logicists mean by "logic". The reason is that, whatever they meant, they certainly meant more than classical logic. <<Nowadays, one can define classical logic as consisting of all those theorems which can be proven in first order languages without the use of nonlogical axioms. We are thus restricting ourselves to first order logic and the use of the deduction rules and logical axioms of such logic. An example of such a theorem is the law of the excluded middle which says that, if p is a proposition, then either p or its negation $\neg p$ is true; in other words, the proposition $p \vee \neg p$ is always true where $\vee$ is the usual symbol for the inclusive "or">> (Snapper, p.1). According to the author, the logicists' definition was more extensive: they had a general concept as to when a proposition should be called a "logical proposition." They stated: *a logical proposition is a proposition that has complete generality and is true in virtue of its form rather than its content* (in Snapper, 1979). Here the word "proposition" is used as synonymous with "theorem." For example, the above law of the excluded middle "$p \vee \neg p$" is a logical proposition. Namely, this law does not hold because of any special content of the proposition p; it does not matter whether p is a proposition of mathematics or physics or what else. The logicists would answer that the proposition holds "because of its form," where by form they mean "syntactical form," the form of "$p \vee \neg p$ being given by the two connectives of everyday speech, the inclusive "or" and the negation "not" (denoted by $\vee$ and $\neg$, respectively). The school failed by about 20% in its effort to give mathematics a firm foundation, since, for example, at least two out of the nine axioms of the formal set theory developed by Zermelo and Fraenkel are not logical propositions in the sense of Logicism; nevertheless, Logicism has been of the greatest importance for the development of modern mathematical logic. In fact, it was Logicism that launched mathematical logic in a serious way. The two quantifiers, the "for all" quantifier $\forall$ and the "there exists" quantifier $\exists$ were introduced into logic by Frege (1970), and the influence of *Principia Mathematica* on the development of mathematical logic is now history.

The philosophy of Logicism is (it is sometimes said), based on the philosophical school called "realism." In medieval philosophy "realism" stands for the Platonic doctrine that states, abstract entities have an existence independent of the human mind. Mathematics is, of course, full of abstract entities such as numbers, functions, sets, etc., and according to Plato all such entities exist outside our mind; the mind can discover them but does not create them. This doctrine has the advantage that one can accept such a concept as "set" without worrying about how the mind can construct a set. According to realism, sets are there to be discovered, not to be constructed, and the same holds true for all other abstract entities. Therefore, realism allows us to accept many more abstract entities in mathematics than a philosophy that limits us to accept only those entities constructed by the human mind. Russell was a realist and accepted the abstract entities that occur in classical mathematics without questioning whether our own minds can construct them.

The school of *Intuitionism* came into being circa 1908, founded by the Dutch mathematician, L. E. J. Brouwer (1881-1966). Logicists simply wanted to show that classical mathematics was a part of logic; intuitionists, on the contrary, felt that there were many things wrong with classical mathematics. By 1908, several paradoxes had arisen in relation to the set theory created by Cantor, started around 1870. The logicists considered these paradoxes as common errors, caused by erring mathematicians and not by a faulty mathematics. The intuitionists, on the other hand, considered these paradoxes as clear indications that classical mathematics itself was far from perfect; from their point of view, mathematics had to be rebuilt from the bottom on up, and that meant starting from the explanation of what the natural numbers 1,2,3…are. According to Intuitionism, all human beings have a primordial intuition for the natural numbers within them; this means that we have an immediate certainty of what is meant by the number 1, and also that the mental process used for the number one can be repeated. Such a repetition allows us to conceptualize the number 2, and so on. In this way human beings can construct any finite initial segment 1,2…n for any natural number *n*. According to Brouwer, the possibility to construct one natural number after the other is given by human beings' awareness of time ("after" refers to time); and his idea comes from the philosopher Immanuel Kant (1724-1804) who already believed in the human beings' immediate awareness of time, and who called such immediate awareness, "intuition," and this is where the name *Intuitionism* (given by Brouwer) comes from.

The first evident difference between *Intuitionism* and *Logicism* is that the intuitionist construction of natural numbers allows one to construct arbitrarily long finite initial segments 1,2…n. It is not possible to construct the whole closed set of all the natural numbers, as has been considered by classical mathematics. Furthermore, the construction of the natural number is both inductive and effective. Inductive, because if we want to construct the number 3, we have to go through all the mental steps have first constructing the 1, and then the 2, and finally the 3: we cannot simply grab the number 3 out of the blue. It is effective, in the sense that, once the construction of a natural number has been finished, that natural number has been constructed in its entirety.

With regards to the intuitionistic definition of mathematics, it should be defined as a mental activity and not as a set of theorems; therefore, "*Mathematics is the mental activity which consists in carrying out constructs one after the other*" (Snapper, p.3); where a *construct* is a mental construction which is inductive and effective (in the sense defined above), and *Intuitionism* maintains that human beings are able to recognize whether a given mental construction contains these two properties. A major consequence of the intuitionistic definition of mathematics is that it is effective or "constructive"; for instance, "*if a real number r occurs in an intuitionistic proof or theorem, it never occurs there merely on grounds of an existence proof. It occurs there because it has been constructed from top to bottom. […] In short, all intuitionistic proofs, theorems, definitions, etc., are entirely constructive*" (Snapper, p.3).

Another consequence of the intuitionistic definition of mathematics is that mathematics cannot be distilled to any other science such as, for example, logic, because such a definition comprises too many mental processes for this kind of simplification. An Intuitionist's attitude toward logic is exactly the opposite of the logicists': the valid part of classical logic is part of mathematics; and any law of classical logic, which is not composed of constructs, is for the intuitionists a meaningless combination of words. For the intuitionists, the classical law of the excluded middle[8] turns out to be a meaningless combination of words. Intuitionists have developed intuitionistic arithmetic algebra, analysis, set theory, etc. However they do not achieve a reconstruction of all classical mathematics, but this does not bother the intuitionists, since their purpose is not to justify all classical mathematics, but to give a valid definition of mathematics and then to "wait and see" what mathematics emerges. Whatever classical mathematics cannot be done in an intuitionistically simple manner is not mathematics for the intuitionist. Therefore, another fundamental difference between Logicism and Intuitionism is that the former wants to justify all of classical mathematics.

The Intuitionistic school represents another crisis in mathematics in the sense that its failure consists in the inability to make intuitionism acceptable to at least the majority of mathematicians. The mathematical community has almost universally rejected intuitionism for three main reasons; the first is that classical mathematicians refuse to reject many theorems because they are meaningless combinations of words for the intuitionists. A second reason comes from theorems that can be proven both classically and intuitionistically. It often happens that the classical proof of such a theorem is short, elegant, and clever, but not constructive. The intuitionists will of course reject such a proof and replace it with their own constructive proof of the same theorem. However, this constructive proof frequently turns out to be about ten times as long as the classical proof and often seems, at least to the classical mathematician, to have lost all of its elegance. An example of this is the fundamental theorem of algebra, which in classical mathematics is proved in about half a page, but takes about ten pages of proof in intuitionistic

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[8] If p is a proposition, then either p or its negation $\neg p$ is true. In other words, the proposition $p \vee \neg p$ is always true.

mathematics. Finally, there are theorems that hold true in intuitionism but are false in classical mathematics. *An example is the intuitionistic theorem, which says that every real-valued function, which is defined for all real numbers, is continuous. This theorem is not as strange as it sounds since it depends on the intuitionistic concept of a function: A real-valued function f is defined in intuitionism for all real numbers only if, for every real number r whose intuitionistic construction has been completed, the real number f(r) can be constructed. Any obviously discontinuous function a classical mathematician may mention does not satisfy this constructive criterion. Even so, theorems such as this one seem so far out to classical mathematicians that they reject any mathematics that accepts them* (Snapper, p.4).

Finally, intuitionism is related to the philosophy called "conceptualism," just as Logicism is related to Realism. Conceptualism maintains that abstract entities exist only insofar as they are constructed by the human mind. Therefore, it can be determined that the abstract entities that occur in mathematics, whether sequences or order-relations, are all constructions of the mind.

German mathematician David Hilbert (1862-1943) founded the *Formalist* in about 1910, even though traces of formalism can be found earlier in nineteenth century since Frege argued against them in the second volume of his *Grundgesetze der Arithmetik* (put the reference). Nevertheless, the modern concept of Formalism, which includes finitary reasoning, must be credited to Hilbert. This last school is much better known than logicism or intuitionism since modern books and courses in mathematical logic usually deal with formalism. It is important not to get confused between *axiomatization* and *formalization*: Euclid axiomatized geometry in about 300 B.C., but formalization only started about 2200 years later with the logicists and formalists. Examples of axiomatized theories are Euclidean plane geometry with the usual Euclidean axioms, arithmetic with the Peano axioms, Zermelo and Fraenkel with the nine axioms, etc.

Hilbert's basic idea was to formalize the various branches of mathematics and then to prove mathematically that each was free of contradictions. Therefore, the original purpose of formalism was to create a mathematical technique by means of which one could prove that mathematics is free of contradictions. The following excerpt attempts to clarify such an idea:

> […] "How do we formalize a given axiomatized theory?"
> Suppose then that some axiomatized theory T is given. Restricting us to first order logic, "to formalize T," means to choose an appropriate first order language for T. The vocabulary of a first order language consists of five items, four of which are always the same and are not dependent on the given theory T. These four items are the following: (1) A list of denumerably many variables – who can talk about mathematics without using variables? (2) Symbols for the connectives of everyday speech, say $\neg$ for "not," $\wedge$ for "and," $\vee$ for the inclusive "or," $\rightarrow$ for "if then," and $\leftrightarrow$ for "if and only if," – who can talk about anything at all without using connectives? (3) The equality sign =; again no one can talk about mathematics without using this sign. (4) The two quantifiers, the "for all" quantifier $\forall$ and the "there exist" quantifier $\exists$; the first one is used to say such things as "all complex numbers have a square root," the second one to say things like "there exist irrational numbers". One can do without some of the above symbols, but there is no reason to go into that. Instead, we turn to the fifth item.
> Since T is an axiomatized theory, it has so called "undefined terms." One has to choose an appropriate symbol for every undefined term of T and these symbols make up the fifth item. For instance, among the undefined terms of plane Euclidean geometry, occur "point," "line," and "incidence," and for each one of them an appropriate symbol must be entered into the vocabulary of the first order language. Among the undefined terms of arithmetic occur "zero," "addition," and "multiplication," and the symbols one chooses for them are of course 0, +, and X, respectively. The easiest theory of all to formalize is Zermelo and Fraenkel set theory since this theory has only one undefined term, namely, the membership relation. One chooses, of course, the usual symbol $\in$ for that relation. These symbols, one for each undefined term of the axiomatized theory T, are often called the "parameters" of the first order language and hence the parameters make up the fifth item.
> Since the parameters are the only symbols in the vocabulary of a first order language, which depend on the given axiomatized theory T, one formalizes T simply by choosing these parameters. Once this choice has been made, the whole theory T has been completely formalized. One can now express in the resulting first order language L not only all axioms, definitions, and theorems of T, but more. One can also express in L all axioms of classical logic and, consequently, also all proofs one uses to prove theorems of T. In short, one can now proceed entirely within L, that is, entirely "formally". (Snapper, p.5)

It is important to underline that both logicists and formalists formalized the various branches of mathematics, but their reasons were totally different. The logicists were interested in formalization to show that the branch of mathematics in question belongs to logic; the formalists wanted to use formalization to prove mathematically that the branch in question is free of contradictions.

Nonetheless, another crisis in mathematics occurred in 1931, when Kurt Gödel showed that formalization couldn't be considered as a mathematical technique by means of which one can prove that mathematics is free of contradictions. Gödel's theorem says, in nontechnical language, "No sentence of L (meant as the first order language L) which can be interpreted as asserting that T is free of contradictions can be proven formally within the language L," therefore mathematics is not able to prove its own freedom of contradictions.

2.5.1.1 The definition of formal proof

The vision of formal proof has taken different shapes during the decades of the twentieth century: from a strict formalist view of proof, to an epistemological interpretation of it, up to a more psychological explanation.

Duval (1991) makes a clear distinction between *argumentation* and *deductive reasoning*. Argumentation is based on the structure of the language and on the listener's representations; therefore the semantic content of the propositions is fundamental. Deductive reasoning is characterized by an "operational status" (*statut opératoire)* given by: 1) Entry propositions (*propositions données),* which are hypotheses or conclusions of a previous step; 2) Rules of inference (*régles d'inférence)*, which are axioms, theorems, and definitions; 3) New propositions (*obtenues)* which are the result of the inference. In a deductive step the propositions are not related to each other for their semantic value, but only by virtue of their operational status.

According to Duval a proof can be so defined only if it is a logical-formal derivation, there is no concern for its semantic value but only for the syntactic value. Any time we talk about the semantic content of a proposition and of the meaning of its enchainment we leave the deductive scheme (the proof) and we enter into the field of argumentation.

A different position is taken by Lolli (1991), who gives value of proof both to a logical-formal derivation and to a more "semantic procedure." In his book "Introduzione alla Logica Formale" he states: *The final knowledge is knowledge about the reliability of the relation of consequence, and they are quite abstract. The work made in process is difficult, if not impossible, to be coded; nevertheless the final proofs which prove the reliability of the relation of consequence sometimes maintain a trace of the involved reflection, and in part they reproduce the informal reasonings* (p.43).

Lolli shows two different approaches to the same proof of Rolle's theorem: If f is a continuous function in [a,b], a < b, and differentiable in (a,b), and f(a) = f(b), exist a point in (a,b) where the first derivative of f is zero.

He compares the logic-formal structure of the proof with a structure containing mixed expressions, with wide use of abbreviations, which correspond to the symbols introduced in the predicative language, especially connectives and quantifiers, but without respecting completely the syntax of the predicative language. He also uses traditional mathematical symbolism, and he puts an asterisk on the main points where argumentative passages have been used and which show a logical relevance.

What follows is an excerpt of the proof of Rolle's theorem through the structure that uses the mixed expressions:

In particular we have already proved that:

<<f continuous in [a,b]>> $\Rightarrow \exists\, x \in [a,b]$ <<f has an absolute maximum in x>>

and therefore, because for hypothesis

<<f continuous in [a,b]>>

we have

(*) $\exists\, x \in [a,b]$ <<f has an absolute maximum in x>>,

We say then

(*) let x_0 be $\in$[a,b] such that <<f has an absolute maximum in x_0>>,

or more schematically

(*) x_0 be $\in$[a,b] and <<f has an absolute maximum in x_0>>,

from which we obtain

(*) $x_0 \in$[a,b]

which we reflect on.

In fact it is not enough, because we want to find a point in (a,b), but we are close to, and then we distinguish:

(*) or $x_0 \in$ (a,b) or (x_0 = a or x_0 = b)

and we treat separately two cases with the idea that in any case we will arrive at the same conclusion.

If $x_0 \in$ (a,b), we prove that f'(x_0) = 0. We consider

(*) $f'(x_0) \neq 0$,

namely, the denial of what we want to obtain. With some algebraic calculus, indicating with Δf the difference quotient of f, not properly because we don't say in which point, but the notation could be expanded, we can see that

$\forall$ x (x $\in$ (a,b) and <<f has a maximum in x>> $\Rightarrow$

<<Δf in an interval of x changes the sign>>),

but

$x_0 \in$ (a,b) and <<f has a maximum in x_0>>

therefore

<<Δf in an interval of x_0 changes the sign>>

while the consideration made, using the theorem of the permanence of the sign comes out also that

<<Δf in an interval of x0 has always the same sign>>

we arrived at a contradiction, therefore we conclude

(*) $f'(x_0) = 0$.

Therefore, we have

$x_0 \in$ (a,b) and <<$f'(x_0) = 0$>>

from which

(*) ∃ x (x ∈ (a,b) and <<f’(x) = 0),

and finally,

(*) x0 ∈ (a,b) ⇒ ∃ x (x ∈ (a,b) and <<f’(x) = 0>>)

And the proof goes on.

This kind of proof is different from a traditional mathematical proof as shown below.

Proof

Because f is continuous on a compact (closed and bounded) interval I = [a, b] it attains its maximum and minimum values.

In case f(a) = f(b) is both the maximum and the minimum, then there is nothing more to say, for then if f is a constant function and f’ = 0 on the whole interval I.

So suppose otherwise, and f attains an extremum in the open interval (a, b), and without loss of generality, let this extremum be a maximum, considering –f in lieu of f as necessary. We claim that at this extremum f(c) we have f’(c) = 0, with a < c <b.

To show this, note that

f(x) – f(c) ≤ 0 ∀x∈ I, because f(c) is the maximum.

By definition of the derivative, we have that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Looking at the one-side limits, we note that

$$R = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

because the numerator in the limit is non-positive in the interval I, yet x - c > 0, as x approaches c from the right.

Similarly,

$$L = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0$$

Since f is differentiable at c, the left and the right limits must coincide, so 0 ≤ L = R ≤ 0, that it is to say, f’(c) = 0. (q.e.d)

2.5.1.2 Lakatos' theory of "Proof and Refutation"

The position taken by Lakatos is critically relevant as it is he who shows that, though mathematics is not an empirical science, its methods are very similar to those of the empirical sciences; he refers to mathematics as *quasi*-empirical. Lakatos goes on to say that mathematics grows through an incessant "improvement of guesses by speculation and criticism, by the logic of proof and refutation" (Lakatos, 1976). In this sense no proof is final, and what leads to the improvement of a proof and its growing acceptance is the social process of negotiation of meaning, rather than the application of formal criteria from the outset.

Lakatos espouses a scheme for the mathematical discovery, namely for the growth of the informal theories of mathematics. It consists of the following stages:

(1) Primitive conjecture.

(2) Proof (a rough thought – experiment or argument, decomposing the primitive conjecture into sub conjectures or lemmas).

(3) 'Global' counterexamples (counterexamples to the primitive conjecture) emerge.

(4) Proof re – examined: the 'guilty lemma' to which the global counterexample is a 'local' counterexample is spotted. This guilty lemma may have previously remained 'hidden' or may have been misidentified. Now it is made explicit, and built into the primitive conjecture as a condition. The theorem – the improved conjecture – supersedes the primitive conjecture with the new proof – generated concept as its paramount new feature[9].

These four stages constitute the kernel of proof analysis. But there are some further standard stages that frequently occur:

(5) Proofs of other theorems are examined to see if the newly found lemma or the new proof – generated concept occurs in them: this concept may be found

[9] *Editor's note:* In other words this method consists (in part) of producing a series of statements $P_1,\ldots, P_n$ such that P_1 &…P_n is supposed to be true of some domain of interesting objects and seems to imply the primitive conjecture C. This may turn out not to be the case – in other words we find cases in which C is false ('global counterexamples') but in which P_1 to P_n hold. This leads to the articulation of a new lemma.

lying at the crossroads of different proofs, and thus come to be of basic importance.

(6) The hitherto accepted consequences of the original (and now refuted) conjecture are checked.

(7) Counterexamples are turned into new examples; new fields of inquiry open up.

The author makes a clear critique of what he calls the "deductive style," defined as that obligatory style of presentation developed by the Euclidean methodology. He claims that such a presentation begins with an accurately formulated list of *axioms, lemmas, and/or definitions.* The axioms and the definitions frequently appear artificial and complicated, and it is never said how such complications arise. The *theorems* follow then the axioms and the definitions. These last ones have heavy conditions; it seems impossible that someone could have ever created such concepts. For each theorem its proof follows.

In the deductive style of presenting a mathematical theory or a mathematical proof, all the propositions are true and all the inferences are valid. Counter examples, refutations, and critiques can never be taken into consideration. Lakatos, states that the deductive style hides the *struggle*, the *adventure*. The whole history disappears, and all the attempts made during the process of proving are neglected and only the final result is dignified.

2.5.1.3 The Debate between Thurston and Jaffe & Quinn

In 1994 William Thurston wrote a very interesting paper (Thurston, 1994) in response to an article by Jaffe and Quinn (1993) who cautioned against weakening the standards of mathematical proof. They advocate two stages through which information about mathematical structures are achieved, distinguishing between *theoretical* mathematics, and *rigorous* mathematics. The former is represented by the phase during which intuitive insights are developed, conjectures are made, and speculative outlines or justifications are suggested. The latter is a proof-oriented phase, where the conjectures and speculations are correct, and are made reliable by proving them. They claim "[…].

The posing of conjectures is the most obvious mathematical activity that does not involve proof.” (p. 6) Furthermore, weak standards of proof cause more difficulty, and they claim:

> Theoretical work should be explicitly acknowledged as theoretical and incomplete; in particular, a major share of credit for the final result must reserved for the rigorous work that validates it (p.10)

To this extent, Thurston’s believes that their article raises interesting issues that mathematicians should pay more attention to, but it also perpetuates some widely held beliefs and attitudes that need to be questioned and examined. According to Thurston, as mathematicians, the correct question to ask is: “How do mathematicians advance human understanding of mathematics?” He also adds: “We [mathematicians] are not trying to meet some abstract production quota of definitions, theorems and proofs. The measure of our success is whether what we do enable people to understand and think more clearly and effectively about mathematics” (p.163).

In such a view, understanding and ways of thinking assume a crucial role: the mathematician should put far greater effort into communicating mathematical ideas, and to accomplish this he needs to pay much more attention to communicating not just his definitions, theorems, and proofs, but also his ways of thinking. There is a need to appreciate the value of different ways of thinking about the same mathematical structure; mathematicians need to focus more energy on understanding and explaining the basic mental infrastructure of mathematics, with consequently less energy on the most recent results. This entails developing mathematical language that is effective for the radical purpose of conveying ideas to people who do not already know them. Jaffe and Quinn’s distinction (1993) regarding *speculation* and *proving* is considered by Thurston as a division that only perpetuates the myth that our progress is measured in units of standard theorems deduced. He goes on to state:

> We have many different ways to understand and many different processes that contribute to our understanding. We will be more satisfied, more productive and happier if we recognize and focus on this. (p. 173)

On the other hand he wants to underline that his stressing the importance of understanding is not any way a criticism of formal proof as such, and remarks:

> I am not advocating the weakening of our community standard of proof; I am trying to describe how the process works. Careful proofs that stand up to scrutiny are very important…Second, I am not criticizing the mathematical study of formal proofs, nor am I criticizing people who put energy into making arguments more explicit and more formal. These are both useful activities that shed new insights on mathematics.” (p.169)

2.5.1.4 Other Contributions to New Interpretations of Proof

Another very interesting contribution to new interpretations of proof has been facilitated by the “computer era.” Computers are employed to create or validate enormously long proofs, some examples of which are the “four-color” theorem (Appel and Haken) or “the solution to the party problem”(Radziszowski and MacKay); such proofs require such long computations that they cannot possibly be performed or verified by a human being. Because computers and computer programs are fallible, mathematicians will have to accept that assertions proved in this way can never be more than provisionally true (Hanna and Yahnke, 1996).

There has been talked of the *zero-knowledge proof* (Blum, 1986), originally defined by Goldwasser, Micali and Rackoff (1985). Such a proof is an interactive protocol involving two parties: a prover and a verifier. It enables the prover to provide to the verifier convincing evidence that a proof exists without disclosing any information about the proof itself. As a result of such an interaction, the verifier is convinced that the theorem in question is true and the prover knows a proof, but the verifier has zero knowledge of the proof itself and is therefore not in a position to convince others.

Hanna and Yahnke (1996) illustrate this concept taking an example from Koblitz (1994):

> Assume a map is colorable with three colors and the prover has a proof, that is, a way of coloring the map so that no two countries with a common boundary have the same color. The prover wants to convince another person that there is a proof (a way of coloring the map) without

> actually revealing it, by letting the other person verify the claim in another way.
> The prover first translates the problem into a graph consisting of vertices (countries) and edges (common boundaries). This means that the prover has a function f: V $\rightarrow$ {R; B; G} that assigns the colors R (red), B (blue), and G (green) to vertices (countries) in such a way that no vertices joined by an edge have the same color. The prover also has two devices: Device A, which sets each vertex to flash a color (R; B; or G), and Device B, which chooses a random permutation of the colors and resets each vertex accordingly. (A permutation might cause all green vertices to switch to blue and all blue vertices to red, for example).
> The interaction between prover and verifier then proceeds as follows. To convince the verifier that there is proof, the prover keeps the colors hidden from the verifier's view, but allows the verifier to grab one edge at a time and see the color displayed at the two ends (the vertices) by Device A. The verifier starts by grabbing any edge, looking at the colors at the ends and noting that they are different. The prover then uses Device B to permute the colors randomly; the permutation is unknown to the verifier. After the permutation, the verifier again grabs any edge and verifies that the colors at the ends are different. The prover again permutes the colors. The two repeat these steps until the verifier is satisfied that the prover knows how to color the map (has a proof)
> This interaction does not tell the verifier how to color the graph, nor does it reveal any other information about the proof. The verifier is convinced that the prover does have a proof, but cannot show it to others. Perhaps the significant feature of the zero-knowledge method, in fact, is that it is entirely at odds with the traditional view of proof as a demonstration open to inspection. This clearly thwarts the exchange of opinion among mathematicians by which a proof has traditionally come to be accepted. (p. 881)

Another innovation introduced by computer scientists in collaboration with mathematicians is represented by *holographic proof* (Cipra, 1993; Babai, 1994). Such a proof consists of transforming a proof into a so-called transparent form that is verified by spot checks, rather than by checking every line. The idea beneath the holographic proof is that it is possible to rewrite a proof (in great detail, using a formal language) in such a way that if there is an error at any point in the original proof it will be spread more or less evenly throughout the rewritten proof (the transparent form). To determine whether the proof is free of error, therefore, one need only check randomly selected lines in the transparent form (Hanna and Yahnke, 1996). By using a computer to increase the number of spot checks, the probability that an erroneous proof will be accepted can be lessened as desired.

All these developments lead practitioners and philosophers of mathematics (Horgan, 1993; Krantz, 1994) to pose intriguing questions: in relation to zero-knowledge and holographic proofs, for example, Babai (1994) asks the following questions: "Are such proofs going to be the way of the future?"; "Do such proofs have a place in mathematics?"; and, "Are we even allowed to call them proofs?" Many others questions have been posed: *Should mathematicians accept mathematical propositions, which have only a high probability of truth, as the equivalent of propositions that are true in the usual sense? If not, what is their status? Should mathematicians accept proofs that cannot be verified by others, or proofs that can be verified only statistically? Can mathematical truths be established by computer graphics and other forms of experimentation? Where should mathematicians draw the line between experimentation and deductive methods?*

Such issues and many other questions are still topics of debate among mathematicians and mathematics educators; for example on the Internet and in the Forum section of *The notices of the American Mathematics Society.* Such debates are a confirmation of the central role that proof still plays in mathematics. Ergo:

> The point we must not lose sight of is that the existence of a new consensus, even one with large remaining areas of disagreement, would not create a situation which would differ in principle from that which has prevailed up to now. […] There has never been a single set of universally accepted criteria for the validity of a mathematical proof. Yet mathematicians have been united in their insistence on the importance of proof [Hanna and Yahnke, 1996; p.884].

2.5.1.5 The role of proof

The incessant debate about what has to be considered a proof, or a formal proof, is accompanied by another very important didactical issue, namely, the role of a proof.
Davis (1986) takes into consideration the role of proof, stating that it may play several different roles. A proof may validate, it may lead to new discoveries, it can be a focus for debate, and it can help eliminate errors. In the real world of mathematicians a proof is never complete and furthermore it cannot be completed:

> There is a view of proof or a view of mathematics which I disagree with and I think is a myth, which says that mathematics is potentially, totally formalizable and, therefore, one can say, in advance, what a proof is, how it should work, etc. (p.336)

Hanna (1990), makes a distinction between *proofs that prove*, and *proofs that explain*, and considers both legitimate proofs, because both meet the requirements for a mathematical proof, namely, they serve to establish the validity of a statement. In each case they consist of statements that are either axioms themselves, or follow from previous statements as a result of the correct application of rules of inference.

A proof that proves shows only that a theorem is true. It is concerned only with substantiation (the proof the truth, validation), and that means, *why-we-hold-it-to-be-so* reasons. Not all proofs have explanatory power; one can even establish the validity of many mathematical assertions by purely syntactic means; with such a syntactic proof one essentially demonstrates that a statement is true without ever showing what mathematical property makes it true. A proof that explains, on the other hand, also shows *why* a theorem is true, and that means, *why-it-is-so* reasons, therefore the term *explain* is used only when the proof reveals and makes use of the mathematical ideas which motivate it.

What follows is Hanna's example that compares a proof that proves, and a proof that explains:

"Prove that the sum of the first n positive integers, S(n), is equal to n(n+1)/2"

A proof that proves

Proof by mathematical induction:

For n=1 the theorem is true.

Assume it is true for any arbitrary K.

Then consider:

$$S(k+1)= S(k)+(k+1)=\frac{n(n+1)}{2}+(n+1)= \frac{(n+1)(n+2)}{2}$$

Therefore the statement is true for k+1 if it is true for k.

By the induction theorem, the statement is true for all n.

Now, this is certainly an acceptable proof: it demonstrates that a mathematical statement is true. What it does not do, however, is show *why* the sum of the first n integers is n(n+1)/2, or what characteristic property of the sum of the first n integers might be responsible for the value n(n+1)/2. (Proofs by mathematical induction are non-explanatory in general).

Gauss's proof of the same statement, however, is explanatory because it uses the property of symmetry (of two different representation of the sum) to show why the statement is true. It makes explicit reference to the symmetry, and it is evident from the proof that its result depends on this property:

A proof that explains

Gauss's proof is as follows:

S= 1+ 2 + 3 +…….+n

S=n+ (n-1)+ (n-2) +…….+1

2S=(n+1)+(n+1)+(n+1)+……+(n+1)=n(n+1)

$$S=\frac{n(n+1)}{2}$$

Another explanatory proof of this same statement is, of course, the geometric representation of the first n integers by an isosceles right triangle of dots; here the characteristic property is the geometrical pattern that compels the truth of the statement. We can represent the sum of the first n integers as triangular numbers (see Figure 1)

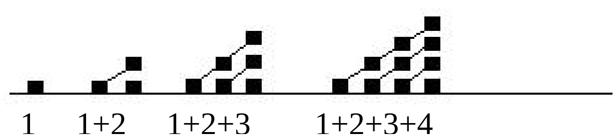


Figure 1: The sum of the first n integers as triangular numbers

The dots form isosceles right triangles containing

S(n)=1+2+3+….+n dots

Two such sums S(n)+S(n) give a square containing n^2 dots and n additional dots because the diagonal of n dots is counted twice. Therefore:

$2S(n)=n^2+n$

$S(n)= \frac{n^2+n}{2} = \frac{n(n+1)}{2}$

Another explanatory proof would be the representation of the first n integers by a staircase-shaped area as follows: a rectangle with sides n and n+1 is divided by a zigzag line (see Figure 2).

The whole area is n (n+1), and the staircase-shaped area, 1+2+3+…+n only half, hence $\frac{n(n+1)}{2}$

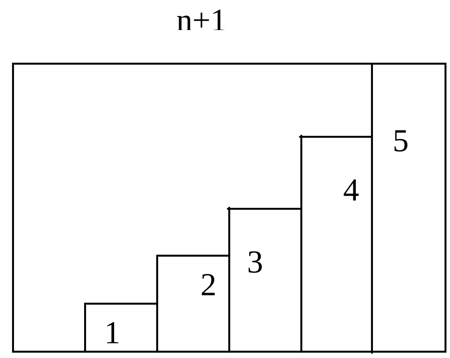


Figure 2: Representation of the first n integers by a staircase-shaped area

Both Gauss's proof and the geometric representation show that one can adopt an explanatory approach to proof in the classroom without abandoning the criteria of legitimate mathematical proof and reverting to reliance on intuition alone. What one must do rather, is to replace one proof of the non-explanatory kind, by another equally legitimate proof that has explanatory power, the power to bring out the mathematical message in the theorem (Hanna, 1990).

Furthermore a proof that convinces need not be a proof that explains: it is certainly possible to be convinced that a statement is true without knowing why it is true. The focus of an explanatory proof is clearly upon understanding, rather than upon deductive mechanism. According to Hanna, understanding is much more than confirming that all the links in a chain of deduction are correct, that in fact the completeness of detail in a formal deduction may obscure rather than enlighten, and that understanding requires some appeal to previous mathematical experience.

Reuben Hersh (1993) distinguishes between the role covered by proof in mathematical research and in the classroom. If we consider the field of mathematical research, then, the purpose of a proof is to convince; in fact, according to the author, the test of whether something is a proof is whether it convinces qualified judges. On the other hand, in the classroom its purpose is to explain. Hersh goes on to say, “Enlightened use of proofs in the mathematics classroom aims to stimulate the students’ understanding, not to meet abstract standards of “rigor” or “honesty”. (p.389)**”**

A further distinction is made between the notion of proof in mathematical practice, namely in the “real life of living mathematicians,” where it is defined as convincing argument, as judged by qualified judges, and formal proof in the sense of formal logic.

Firstly, formal proof can exist only within a formalized theory. Formal proof has to be expressed in a formal vocabulary, founded on a set of formal axioms, reasoned about by formal rules of inference. But the passage from an informal, intuitive theory to a formalized theory inevitably entails some loss or change of meaning. Consequently, any result that is formally proved may be challenged: “How faithful is this statement and proof to the informal concepts we are actually interested in?”

Secondly, for many mathematical investigations, full formalization and complete formal proof, even if possible in principle, may be impossible in practice. These proofs may require time, patience, and interest beyond the capacity of most mathematicians.

Very often in journal and textbooks proof functions as the last judgment, the final word before a problem is put to bed. But the essential mathematical activity is finding the proof, not checking after the fact that it is indeed a proof. At the stage of creation, proofs are often presented in front of a blackboard, hopefully and tentatively. The detection of an error or omission is welcomed as a step toward the improvement of the proof.

Hardy (1929), one of the most eminent English mathematicians of his day, wrote:

> I have myself always thought of a mathematician as in the first instance an observer, who gazes at a distant range of mountains and notes down his observations. His object is simply to distinguish clearly and notify to others as many different peaks as he can. There are some peaks, which he can distinguish easily, while others are less clear. He sees A sharply, while of B he can obtain only transitory glimpses. At last he makes out a ridge which leads from A and, following it to its end, he discovers that it culminates in B. B is now fixed in his vision, and from

> this point he can proceed to further discoveries. In other cases perhaps he can distinguish a ridge, which vanishes in the distance, and conjectures that it leads to a peak in the clouds or below the horizon. But when he sees a peak, he believes that it is there simply because he sees it. If he wishes someone else to see it, he points to it, either directly or through the chain of summits that led him to recognize it himself. When his pupil also sees it, the research, the argument, the proof is finished. (p.18)

Hersh, recalling Hardy's words claims that all real life proofs are to some degree informal. The formal logic picture of proof is not a truthful picture of real-life mathematical proof, to this extent he perlustrates three different meanings of *proof*:

1) As the English word "prove" it means: test, try out, and determine the true state of affairs.

2) In mathematics, "proof" has two meanings, one in common practice; the other specialized in mathematical logic and in philosophy of mathematics:

The first one, the "working" meaning is:

> An argument that convinces specialized judges

The second mathematical meaning, the "logic" one, is:

> A sequence of transformations of formal sentences, carried out according to the rules of the predicate calculus.

Related to the role of proof in classroom, Hersh claims it is not to convince; convincing is no problem. Students are all too easily convinced. What a proof should do for the student is to provide insight into why a theorem is true. He categorizes two opposing views on the role of proof in teaching: he defines them as *Absolutism* and *Humanism.*

The Absolutist view is characterized by the idea that "without complete, correct proof, there can be no mathematics." Mathematics in such an approach is seen as a system of absolute truths independent of human construction or knowledge; therefore, mathematical proofs are external and eternal. Proofs are to admire, hopefully to understand, but not to play with, not to break apart. Hersh defines the figure of the Absolutist teacher as the one who tells the student nothing except what he will prove (or assign to the student to prove). The proof chosen will be either the most general, or the shortest. He will not be concerned about how explanatory the proof is, because explanation is not the purpose of the proof, but it is certification: admission into the catalog of primarily absolute truths.

In the Humanist view "Proof is complete explanation." Proofs are not obligatory rituals. In brief, the purpose of proof is – understanding. The choice of whether to present a proof *as is*, to elaborate it, or to abbreviate it, depends on which is likeliest to increase the student's understanding of concepts, methods, and applications.

2.5.2. Proof as process

2.5.2.1. Harel's Theory of Proof Schemes

Harel (1998) defines the *process of proving* in the following way:

> *By "proving" we mean the process employed by an individual to remove or create doubts about the truth of an observation. (p. 241)*

Ascertaining and *persuading* are the two sub processes included in the process of proving:

> *Ascertaining* is the process an individual employs to remove her or his own doubts. *Persuading* is the process an individual employs to remove others' doubts about the truth of an observation. (p. 241)

One of the main concerns of Harel's research is to understand and describe how individuals prove or justify, more specifically, how students ascertain for themselves or persuade others of the truth of a mathematical observation. To this extent a classification of *Proof Schemes* has been created, where the proof scheme has been defined as: *A person's proof scheme consisting of what constitutes ascertaining and persuading for that person* (p.244).

The author stresses that the definitions of the process of proving and proof scheme are deliberately psychological and student-centered; each of the categories of the proof schemes in the classification represents a cognitive stage, an intellectual ability, in students' mathematical development, and all have been derived from the observations of the actions taken by actual students in their process of proving.

What characterizes the construction of the proof schemes is the individual's scheme of doubts, truths, and convictions, in a given social context. The entire system is constituted by three main categories of proof schemes, each of them containing several subcategories. The first category is represented by the *External conviction Proof Schemes:* "When the formality in mathematics is emphasized prematurely, students come to believe that ritual and form constitute mathematical justification. When students merely follow formulas to solve problems, they learn that memorization of prescriptions, rather than creativity and discovery, guarantee success. And when the teacher is the sole source of knowledge, students are unlikely to gain confidence in their ability to create mathematics…schemes by which doubts are removed by a) the ritual of the argument – the ritual proof scheme; b) the word of an authority – the authoritarian proof scheme; or c) the symbolic form of the argument - the symbolic proof scheme" (p.246).

The second category is represented by the *Empirical Proof Schemes:* "In an empirical proof scheme, conjectures are validated, impugned or submitted by appeals to physical facts or sensory experiences" (p.252).

In the Empirical Proof Scheme it is possible to distinguish between two kinds of schemes: The *inductive* empirical proof scheme and the *perceptual* empirical proof scheme. A person possesses an *inductive* proof scheme when they ascertain for themselves and persuade others about the truth of a conjecture by *quantitatively evaluating*[10] the conjecture in *one or more* specific cases. The *perceptual* proof scheme is characterized by perceptual observations made by means of rudimentary mental images – images that consist of perceptions and a coordination of perceptions, but lack the ability to transform or to anticipate the results of a transformation, "The important characteristic of rudimentary mental images is that they *ignore transformations on objects or are incapable of anticipating results of transformations completely or accurately"* (p.255).

The third category is represented by the *Analytical Proof Schemes;* in this case the conjectures are validated by means of logical deductions. By logical deduction is meant much more than what it is commonly referred to as the "method of mathematical demonstration" – a procedure involving a sequence of statements deduced progressively by certain logical rules from a set of statements accepted without proofs (i.e., a set of axioms).

[10] e.g., direct measurements of quantities, numerical computations, substitutions of specific numbers in algebraic expressions, etc.

Two subcategories belong to this category are: the *transformational* proof scheme, and the *axiomatic* proof scheme.

The *Transformational* proof scheme belongs to the third category and it is so defined: “Transformational observations involve operations on objects and anticipations of the operations’ results. They are called transformational because they involve transformations of images- perhaps expressed in verbal or written statements- by means of deduction” (p.258). The following episode is an example of transformational proof scheme:

> Amy demonstrates to the whole class how she imagines the theorem, “The sum of the measures of the interior angles in a triangle is 180°.” Amy says something to the effect that she imagines the two sides AB and AC of a triangle ABC being rotated in opposite directions through the vertices B and C, respectively, until their angles with the segment BC are 90° (Figure 3a, b). This action transforms the triangle ABC into the figure A’BCA,’’ where A’B and A’’C are perpendicular to the segment BC. To recreate the original triangle, the segments A’B and A”C are tilted toward each other until the points A’ and A” merge back into the point (Figure 3c). Amy indicates that in doing so she “lost two pieces” from the 90° angles B and C (i.e. angles A’BA and A’’CA) but at the same time “gained these pieces back” in creating the angle A. This can be better seen if we draw AO perpendicular to BC: angles A’BA and A”CA are congruent to angles BAO and AOC, respectively (Figure 3d)

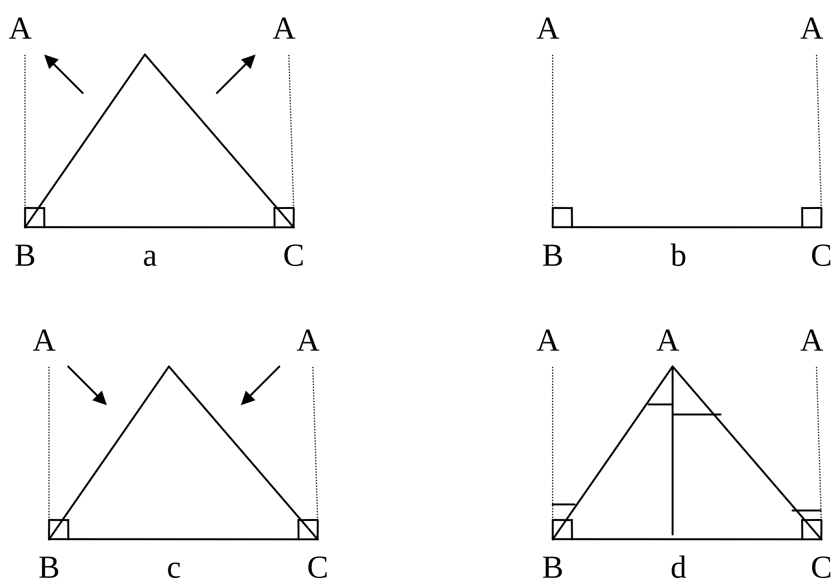


Figure 3: Amy's dynamic representation

The interpretation given by Harel is the following: Amy views a triangle as a dynamic entity; it is a product of her own imaginative construction, not of a passive perception. Her operations were goal oriented and intended the generality aspect of the conjecture. She transformed the triangle and was fully able to anticipate the results of the transformations, namely, that the change in the 90° angles B and C caused by the transformations is compensated for by the creation of the angle A. All this leads to her deduction that the sum of the measures of the angles of the triangle is 180°.

The Transformational Proof Scheme possesses two different cognitive levels: the *internalized* proof scheme, and the *interiorized* proof scheme.

According to Harel, “An internalized proof scheme is a transformational proof scheme that has been encapsulated into a proof heuristic – a method (of proof) that renders conjectures into facts” (p. 262). Harel gives the following example: “to prove two segments in a given figure are congruent, students commonly look for two congruent triangles that respectively include the two segments” (p. 262). Such a proof heuristic is abstracted by the students from the repeated application of an approach they have often found to be successful.

Harel continues, “An interiorized proof scheme is an internalized proof scheme that has been reflected upon by the person possessing it so that they become aware of it. A person’s awareness of the proof scheme is usually observed when the person describes it to others, compares it to other proof schemes, specifies when it can or cannot be used…by definition, the interiorization process cannot occur unless the internalization process has taken place (p. 265).

The last component of the Analytical Proof Scheme is the *Axiomatic* Proof Scheme, which can be Intuitive, Structural, or Axiomatizing. Harel determines, “When a person understands that at least in principle a mathematical justification must have started originally from undefined terms and axioms (facts, or statements accepted without proof), we say that person possesses an *axiomatic* proof scheme” (p.273).

The *intuitive-axiomatic* proof scheme is possessed by a person who is necessarily aware of the distinction between the undefined terms, such as “point” and “line,” and defined terms, such as “square” and “circle,” and between statements accepted without proof, and ones that are deducible from other statements. However, Harel clarifies: “He or she, however, may be able to handle only axioms that correspond to her or his intuition, or ideas of self-evidence, such as *for any a and b in F, a + b = b + a* in relation to her or his experience with real numbers, or *one and only one line goes through two points* in relation to her or his imaginative space” (p.273).

A *structural* proof scheme is an axiomatic proof scheme by which one thinks of conjectures and theorems as representations of situations from *different* realizations that are understood to share a common structure characterized by a collection of axioms.

According to the author the structural proof scheme is a cognitive prerequisite to the *axiomatizing* proof scheme – a scheme by which a person is able to investigate the implications of varying a set of axioms, or to axiomatize a certain field.

2.5.2.2 Cognitive Unity of Theorems

The definition of Cognitive Unity was born as product of a study concerning the difficulties met by the students in the approach to proof. When confronted with students’ statement of “empty mind” when they face a proof, Boero along with other researchers underline the importance that beginners’ proving must be rooted in the argumentative activity consisting in the search and elaboration of arguments for the plausibility of the conjecture (see Boero, Garuti & Mariotti, 1996; Garuti, Boero, Lemut & Mariotti, 1996; Mariotti et al., 1997; Boero, Garuti & Lemut, 1999).

The implications for the research are:

- Identification of possible kinds of inference intervening in the conjecturing process and their roots (within the school and outside of the school);
- Investigation about possible links between the identified kinds of inference during the conjecturing phase, and strategies during the subsequent proving phase, in particular as this concerns the classic “analysis” and “synthesis” methods.

In such a context the “cognitive unity of theorems” (Garuti et al., 1996) has been defined as *that peculiar situation where some arguments, produced for the plausibility of the* *conjecture during the conjecture production (or appropriation) phase, become ingredients for the construction of proof.*

According to Garuti's initial intuition, cognitive unity of theorems concerns the possible continuity between some aspects of the conjecturing process and some aspects of the proving process: first of all, the arguments. During the conjecturing phase some relations, known properties, evidences, general rules, etc. can be produced or evoked as reasons for the plausibility of the conjecture. Some of these arguments can intervene in the proving process as relevant arguments to support further findings (e.g. generalizations), or as components of the final deductive reasoning. But cognitive unity (as "that particular situation…) concerns also those conditions that allow some arguments, produced during the conjecturing phase, to be exploited during the proving phase.

The cognitive unity is characterized by:

- Continuity of the mathematical frame (if this continuity is not kept, most arguments produced in the conjecturing phase are not recyclable in the proving phase: consider conjecturing within a synthetic geometry frame and proving within an analytic geometry frame).
- The continuity of the exploration strategies and heuristics (if this continuity is not kept, arguments which are relevant in a given exploration during the conjecturing phase may become useless, or even be forgotten, in another kind of exploration during the proving phase).
- The continuity of the external representation (an important change in the external representation between the conjecturing phase and the proving phase can make unavailable all the arguments that are strictly related to a peculiar representation – for instance, visual arguments related to graphs of functions can become unavailable when we move to use algebraic language).

It is important to clarify that "cognitive unity of the theorems" concerns the arguments and related conditions of continuity in the transition from conjecturing to proving, not the possible "structural" analogy, or "continuity" between the argumentation during the different phases of the activity that springs from the search of a conjecture to the text of the proof.

The problem from a structural point of view has been faced by Pedemonte (2002) who distinguishes between "reference system continuity" (the one considered by Garuti et al., 1996; what has been defined as "cognitive unity of theorems") and the "structural continuity" concerning the structure of argumentation, according to Toulmin's model. In particular, very frequently it happens that the "reference system continuity" is kept, while the "structural continuity" is broken. The following example shows such a situation:

> The example concerns an abductive process in the phase of the conjecturing and/or early proof construction. In this case, it is evident that the search for a general condition under which the regularity under scrutiny is a possible consequence, is guided by the need of providing a theoretic argument in the perspective that it becomes a premise for a deductive step.
> As we will see in the protocol, the break in structural continuity consists of the shift
> …from a creative process:
> given an argument B (an experienced regularity)
> both an argument A (a condition for regularity)
> and a possible inference A $\rightarrow$ B are searched for.
> …to a deductive enchaining:
> if A, then B, because…
> The shift implies inverse temporal movements between B and A in the two phases of the activity:
> From B as a possible consequence, to A as a possible case;
> Then,
> From A to B through deduction.
> Structural continuity is broken as a consequence of a cultural need, which ensures both the abductive search for arguments and their final deductive enchaining. We can remark that cognitive unity (as "reference system continuity") is kept.
> Here there is an exemplary protocol illustrating these phenomena.
> Undergraduate mathematics students were asked to produce (and prove) a conjecture that generalizes the elementary theorem: "The sum of two consecutive odd numbers is divisible by four." About one half of students produced the conjecture "The sum of an even number P of consecutive odd numbers is divisible by 2P." Only one third of them produced a valid, rigorous proof for this conjecture (cfr. Boero et al., 2002).
> Elena (a clever student) performs the algebraic proof of the given theorem "(2K+1) + (2K+3) = 4K + 4 = 4 (K+1)"; then she writes:

> OK, this formally proves that the theorem that I must generalize is true, but it does not explain why it is really true. It I want to get a generalization, I think that I must try to understand better why it is true.
> 3+5=8
> 7+9=16, double of eight
> 11+13=24 double of 12;
> 111+113=124
> 19+21=40
> 109+111=220
> Two odd consecutive numbers…I consider the even number between them.
> (2s-1) is the preceding odd number
> (2s+1) is the following number.
> Now I understand: the sum is 4s.
> This might be a particular case of the sum of
> …+2s-3+2s-1 + 2s+1 +2s+3+…
> A sum of couples of odd numbers…it should make 4s multiplied for the number of couples, the mechanism of cancellation is the same! Strong: I notice that it is like the anecdote about the young Gauss!
> (The anecdote had been discussed during the course, following M. Wertheimer's interpretation in his book "Productive thinking").
> The conjecture is: 'The sum of an even number of consecutive odd numbers is divisible by the double of the number of added numbers.'
> The proof might be:
> I represent the sum of couples of odd numbers as balanced couples "even intermediate – number, even intermediate + number."
> Their sum repeats the double of the intermediate even number as many times as the number of couples (underlining in the original). Then it is true that the sum is divisible by the double of the even number of couples.

2.5.3. The teaching of proof

The great changes in the views and interpretations of proof and its role have also influenced the approach to teaching it. The late 50's were characterized by the entrance of the "New Math", influenced by the work of Bourbaki; until that moment the teaching of proof in mathematics education was limited to geometry, where a proof was more a ritual to be followed than a source of deeper mathematical understanding. The New Math-influenced mathematics curriculum introduced a new emphasis on axiomatic structure and proof, and was seen going well beyond geometry: this reform, like others, aimed at the improvement of mathematical understanding. Unfortunately, the "new math" failed the goal concerning proof; its demise was due to an exaggerated emphasis on formal proof.

After this decline, several other approaches emerged: from instruction by discovery to cooperative learning, learning through problem –solving to classroom interaction. All of these approaches exercised significant influence on the curriculum even though none of them gained universal acceptance.

Among the most influential theories of mathematics education we may recall *Constructivism* in its various forms. The basic tenet of this approach is that knowledge cannot be transmitted, but must be constructed by the learner (von Glasersfeld, 1983; Cobb, 1988; Kieren and Steffe, 1994). Such a theory has been at the center of several misinterpretations; many have seen in it an approach that undermines the role of the teacher in the classroom; at precisely the same time a number of experimental studies have just confirmed the importance of the role of the teacher. These studies have shown the value of approaches such as debating, restructuring, and pre-formal presentation, all of which posit a crucial role for the teacher in helping students to identify the structure of a proof, to present arguments, and to distinguish between correct and incorrect arguments.

In tandem with approaches to teaching proof through classroom debate, we may utilize Alibert (1988), who designed an experimental study in which teachers had students engage in debate to assist in the understanding of a mathematical justification. His concern is about mathematical productions of many students at the beginning of the first year in the university, who often seem to mimic the writing of the teacher. According to Alibert, syntactic characteristics often seem to prevail over semantic characteristics; therefore the control of the meaning does not appear to be a primary purpose of the students' texts.

What the author wants to underline is that meanings are not used as a means of controlling the results of algorithms. During his observations he noticed that for the students proof is usually only a formal exercise to be completed for the teacher, but there is no deep necessity for it; to this extent he designed an experimental teaching method applied to teach mathematics in the first year at the University, and set in place a particular theoretical framework. This framework is based on

a) “théorie des situations didactiques” (Brousseau, 1986); b) plurality of conceptual settings (Douady, 1986); c) the development of a sense of the need for proof generated by the role of contradictions (Balacheff, 1982); d) the importance of the role of the group of students for the construction of meaning (Bishop, 1985; Balacheff & Laborde, 1985); e) meta-mathematical factors, such as systems of representation in mathematics, and the way mathematics is learned as very important tools in the learning process (Schoenfeld, 1983); and f) the constitution of a “learner’s epistemology,” meant as the set of problems and situations the single student builds during the constitution process of a particular concept.

The author believes that the necessity and the functionality of proof can only surface in situations in which the students meet uncertainty about the truth of mathematical propositions. According to this idea the generation of scientific debate and how it unfolds in the classroom takes place as follows:

First step: The teacher initiates and organizes the production of scientific statements by the students. These are written on the blackboard without any immediate evaluation of their validity.

Second step: The statements are put to the students for consideration and discussion. They must come to decisions about their validity by taking a vote; each opinion must be supported in some way, by scientific argument, by proof, by refutation, by counter-example…

Third step: The statements that can be validated by a full demonstration to become theorems; those found to be incorrect are preserved as “false statements” associated with appropriate counter-examples. The students’ lecture notes are observed to contain these two kinds of statements. (p.32)
(For detailed examples, see Alibert, 1988; p.32)

In this form of “scientific debate” the proof arguments made by the students are not addressed to the teacher but to the other students; and proofs are distinguished between *proofs to convince* and *proofs to show.* In the former, arguments are produced to convince someone (such as another student) of something that is not already a part of his or her institutionalized knowledge; in the latter the target is to show someone (such as the teacher) that we have reached some knowledge that he already possesses; the activity involved in the first process is fundamentally different from the one involved in the second, in such a way that it is able to produce a deepening of knowledge and its meaning. In this situation the student, therefore, tries to convince others, and himself or herself at the same time, of the truth of a conjecture that has been formulated (by other students or by him/herself) in answer to some problem the whole group of students is trying to solve; they all know that the conjecture is not necessarily true, and in particular that it is not yet established as an item of institutionalized knowledge ('co-didactic situation'). This process of interactions and conflicts between the students' conception will enhance the need for clarifications of contradictions leading to an emerging need for proof (See Alibert, 1988, for further details).

Other researchers have investigated the use of classroom debate in order to teach proof and its uses. Balacheff (1988) talks about 3-stage method (débat socio-cognitif): in such a procedure the teacher guides the students through discussions in which they come up with a conjecture, perform appropriate measurements to test it, and then create a proof in support of their conjecture.

Further studies have focused on the meaningfulness of a proof, namely, Movshovitz-Hadar (1988) and Leron (1983) have shown that there are a number of techniques to make proofs more meaningful to students. For example, the same theorem may be proved in several different ways in the same class; or a proof can be restructured to make its overall structure clear, before each step is looked at in detail. A proof by contradiction can be avoided, when possible, replacing it with a constructive one.

Movshovitz-Hadar insists on the importance of the fact that the more stimulating a presentation of a theorem is the more successful is the setting of the stage for the coming proof:

> Very often in going through a formal proof, particularly those suffering from the "let us define a function" syndrome (Avital, 1973) the student feels treated shabbily. The origin of the proof remains a mystery and the student is left with a frustrated feeling of not being wise enough, not only not as wise as the person who invented the proof, but not even wise enough to understand how the inventor came up with the idea. The attitude towards mathematics, which is encouraged this way, is: "I'll never understand it, it is not for me" (p.18)

In Movshovitz-Hadar's article (1988), she presents two theorems and several ways to present them and their proofs. The basic point is to trigger students' intellectual curiosity in order to make them wonder: "HOW COME???" in reading or hearing a statement, in contraposition with a common reaction translated by "SO WHAT???" In this sense if mathematics teachers agree to give first priority to thought-provoking presentations, priority should be given to the ones causing some kind of surprise.

Concerning this argument I believe it is beneficial to examine one of the examples given by the author, regarding a property of prime numbers. The issue is introduced by Movshovitz-Hadar through what she defines *a surprising imposition*. She writes:

<<Honsberger (1970) tell us that Sundaram's Sieve was invented in 1934 by a young East Indian student, named Sundaram, as an instrument for sifting prime numbers from positive integers. The Sieve consists of the infinite table represented by Table 1.

Table 1: Sundaram's Sieve

4	7	10	13	16	19	22	25	…
7	12	17	22	27	32	37	42	…
10	17	24	31	38	45	52	59	…
13	22	31	40	49	58	67	76	…
16	27	38	49	60	71	82	93	…
.	.	.	.	.	.	.	.	…

The remarkable property of this table is: If N occurs in the table, then 2N+1 is not a prime number; if N does not occur in the table, then 2N+1 is a prime number.>> (Ibid. p.75).

It is surprising because even though the entries in the table have immediately visible additive properties, they do not have anything that ties them with primality; basically a multiplicative property. As suggested by Mason et al. (1985), mathematical thinking is provoked by a gap between new impressions acting on old views (p.15); in the prime number case, it is the gap between a collection of arithmetic progressions having an obvious regularity resulting from their additive property, and the fact that prime numbers are defined by a multiplicative property and are known to have little regularity.

Three different presentations of the proof are taken into consideration:

A formal proof

In Table 1 the first row comprises all the terms of the infinite progression beginning with 4, 7, 10… This progression is also used to generate the first column. Succeeding rows are then completed so that each consists of an arithmetic progression, such that the common differences in successive rows are the odd integers 3, 5, 7, 9, 11… Sundaram's claim is: If the number N occur in this table, then 2N+1 is not a prime number; if N does not occur in the table, then 2N+1 is a prime number. Honsberger (Ibid. pp.84-5) proceeds with the proof as follows:

> Proof: We begin by finding a formula for the entries in the table. The first number in the nth row is
>
> $4+(n-1)3 = 3n+1$
>
> The common difference of the arithmetic progression comprising the nth row is 2n+1; hence the mth number of the nth row is
>
> $3n+1+(m-1)(2n+1) = (2m+1)n + m.$
>
> Now, if N occurs in the table, then $N = (2m+1)n + m$ for some pair of integers m and n. Therefore,
>
> $2N+1 = 2(2m+1)n+2m+1 = (2m+1)(2n+1)$
>
> is composite.
>
> Next, we must show that, if N is not in the table, 2N+1 is prime; or, equivalently, if 2N+1 is not prime, N is in the table. So, suppose 2N+1 = ab, where a, b, are integers greater than 1. Since 2N+1 is odd a and b must both be odd, say
>
> $a=2p+1$, $b=2q+1$
>
> so that
>
> $2N+1=ab=(2p+1)(2q+1)=2p(2q+1)+2q+1$
>
> and
>
> $N=(2q+1)p + q.$
>
> But this means N appears as the qth number of the pth row in the table. We conclude that N occurs in the infinite table represented by the Table 2 if 2N+1 is not a prime number.

Movshovitz-Hadar declares that every step in this proof is clear; Sundaram's sieve is admittedly valid, and yet the manner in which the Indian student came up with his remarkable idea remains altogether mysterious. Probably, many readers feel a bit disappointed after going carefully through this proof for we still have no answer to the question, what do these arithmetic progression have to do with primality? This proof does not make us any wiser. The tension caused by the surprising declarative statement is not relieved (This is a proof that Hanna would define a proof as one that proves, not that explains).

Let us continue with the second approach.

A gap-bridging proof

1. The claim we wish to prove concerns the odd numbers. Let us transform every number N occurring in Table 1 to the corresponding K satisfying K=2N+1, as shown in Table 2. Consequently, the statement to be proved becomes: K occurs in the infinite table represented by Table 2 if K is not prime.

Table 2: Sundaram’s Sieve Transformed

9	15	21	27	33	39	45	51	…
15	25	35	45	55	65	75	85	…
21	35	49	63	77	91	105	119	…
27	45	63	81	99	117	135	153	…
33	55	77	99	121	143	165	187	…
.	.	.	.	.	.	.	.	…

2 As any odd integer is a product of two odd integers, the infinite multiplication table of all pairs of odd integers (Table 3), must contain all primes except 2.

3 By definition, *all* prime numbers (except 2) occur in the first row and column of this table, and *no* prime number occurs elsewhere. In addition, any odd composite integer must occur at least once outside of the first row and column.

4 On the other hand, if we omit the first row and the first column of Table 3, the remainder is identical with Table 2. This is because, like any integer-multiplication table, Table 3 is, in fact, a set of row arithmetic progressions with the marginal numbers as their respective common differences.

5 We conclude that Table 2 contains *all* odd composite integers and *no* primes. In other words: For any odd integer *K*, if *K* is prime, then it does not occur in Table 2, and if *K* does not occur in Table 3, then *K* is prime. (Q.E.D.)

Table 3: Multiplication table of odd integers

X	1	3	5	7	9	11	13	15	17	…
1	1	3	5	7	9	11	13	15	17	…
3	3	9	15	21	27	33	39	45	51	…
5	5	15	25	35	45	55	65	75	85	…
7	7	21	35	49	63	77	91	105	119	…
9	9	27	45	63	81	99	117	135	153	…
11	11	33	55	77	99	121	143	165	187	…
.	.	.	.	.	.	.	.	.	.	…

This proof bridges the gap, created by the statement of the theorem, between the arithmetic progressions and prime numbers. The bridging takes place at the multiplication table of odd numbers in step 4 where arithmetic progressions and primes intersect. This proof may be called "responsive" since it responds to the stimulation created by the theorem. In general, responsive proofs usually leave most of the audience with an appreciation of the invention, along with a feeling of becoming wiser.

The last one considered by the author is:

A bottom-up development of the proof (and of the theorem)

In this case we suppose we have no idea whatsoever about Sundaram's Sieve. We will consider the mainline of a sequence of tasks leading gradually to the discovery of the sieve.

The goal: At the end of this sequence you will have discovered an algorithm separating all primes from the positive integers.

a) Recall the definition of a (natural) prime number and a (natural) composite number.

b) What property do all prime numbers except 2 have in common? (Answer: All are odd).

c) In view of the previous finding, how can the goal be simplified? (Answer: In order to separate the primes from the positive integers it is sufficient to separate the odd-primes from the odd-positive integers).

d) Construct the multiplication table of the odd positive integers up to 17 (Result: See table 3 above).

Consider this table as representative of the infinite table of the products of all pairs of odd integers. Study its properties:

1. What property do all entries in the infinite table have in common? (Answer: All are odd integers as any odd number is a product of at least one pair of odd numbers).
2. Where do *all* prime numbers occur in the infinite table? (Answer: In the first row and column).
3. Where do *only* composite numbers occur? (Answer: In the complementary part of the table, that is in all but the first row and column).

e) If we omit the first row and the first column of the infinite multiplication table of odd positive integers, what kind of integers are left in the reduced table? (Answer: The reduced table contains *all* composite odd numbers and *only* them).

f) Restate your findings in terms of a conditional statement:

If an odd integer K occurs in the reduced table, then…

If and odd integer K does not occur in the reduced table, then…

g) Let K designate any odd integer, then $K=2N+1$ for some integer N. Transform the reduced table by replacing K with the corresponding N and restate your summary in term of N. (Ans.: The transformed table coincides with Table 1 and above the statement is Sundaram's: If N occurs in the table, then $2N+1$ is not a prime and vice versa.)

h) Based upon the finding in step "g" create a flow chart describing an algorithm by which you can now determine for any given positive integer N whether or not N is prime (Result: For an elaboration of this task see Hadar & Hadass, 1983).

Clearly, this task-sequence proceeds in a bottom-up fashion, from previous knowledge about prime numbers and odd integers, to the discovery of Sundaram's Sieve. It is noteworthy that the theorem is stated at the end of the process, at which stage it has already been proven. The sequence, therefore, is constructive (p.16).

The “structural method” (Leron, 1983) is an alternative method to the original step-by step, “linear” way of presenting proofs that proceeds unidirectionally from hypotheses to conclusions, considered by Leron as probably well suited for securing the validity of proofs, but unsuitable for mathematical communication. The aforementioned method aims to increase the comprehensibility of mathematical presentations while retaining their rigor; and its basic idea is to arrange proofs in levels, going from the top to the bottom. Each level consists of short autonomous “modules,” and each module embodies one major idea of the proof.

The top level gives a precise, but in very general terms, main line of the proof; the second level elaborates on the generalities of the top level, supplying proofs for unsubstantiated statements, details for general descriptions, specific constructions for objects whose existence has been merely asserted and so on; the procedure continues down where each level supplies more details. Leron writes: “One may think of the structural approach as viewing proof (which is at ground level) from a tall building. When viewing from the top we see the whole proof at a glance, but only in vague outline, no details can be discerned. As we descend the levels of the building, a zooming effect occurs: our view encompasses smaller and smaller segments of the proof, but these are seen with more and more clarity” (p.175).

The following pictures represent Leron’s pictorial comparison of the two approaches: the linear method is represented by an oriented line segment; the structural method by a “structure diagram.”

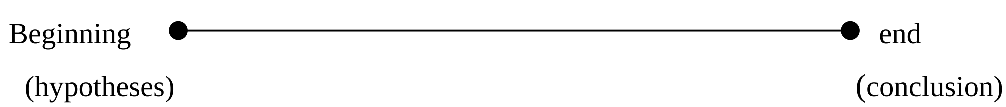


Figure 4: The linear method

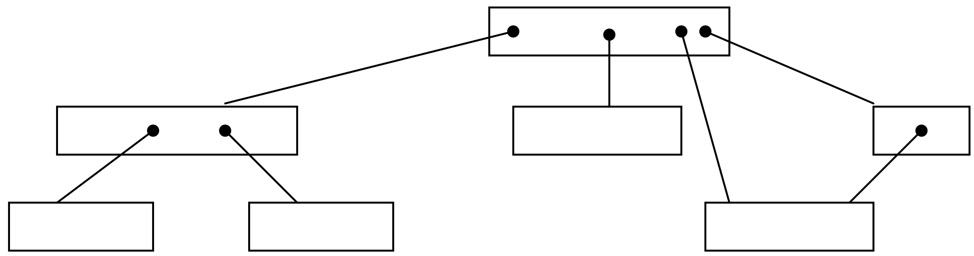


Figure 5: The structural method

The top level is normally very short and free of technical details (i.e., notational, computational, etc.,); the bottom level is quite detailed, resembling in this respect the standard linear proof. The intermediate levels entail the role of facilitating a smooth transition from the generalities of the top level to the details of the bottom, from the global to the local perspective; and in each module (box) the argument flows linearly, but it is very short and "flat," therefore it can again be grasped at a glance.

What follows is an example chosen from amongst the several ones given by Leron concerning Algebra, Calculus, Geometry, Linear Algebra and so on; it compares the two aforementioned structures.

A Theorem on Limits

Theorem: *if* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$, *then* $\lim_{x\to a} f(x)\, g(x)= LM$.

Proof in the linear style (taken from a real calculus textbook).

Let $\varepsilon > 0$ be given and let η be the smaller of $\sqrt{\varepsilon/3}$ and $\varepsilon/3(1+|L|+|M|)$. Since $\lim_{x\to a} f(x) = L$, there exists a $\delta_1 > 0$ such that $|f(x)-L| < \eta$ whenever $0<|x-a|< \delta_1$. Similarly, there exists a $\delta_2 >0$ such that $|g(x)-M|< \eta$ whenever $0<|x-a|< \delta_2$. Let δ be the smaller of δ_1 and δ_2. Now if $0<|x-a|< \delta$, then $0<|x-a|<\delta_i$, i=1,2 and so we have:

$$|f(x)g(x)-LM| = |L(g(x)-M) +M(f(x)-L)+(f(x)-L)(g(x)-M)| \leq |L|\,|(g(x)-M)| + |M|\,|(f(x)-L)| + |(f(x)-L)|\,|(g(x)-M)| <$$

$$|L|\,\varepsilon/3(1+|L|+|M|)+|M|\,\varepsilon/3(1+|L|+|M|)+\sqrt{\varepsilon/3}\,\sqrt{\varepsilon/3} \leq \varepsilon/3\cdot\varepsilon/3\cdot\varepsilon/3 = \varepsilon \quad \text{(q.e.d.)}$$

Fortunately, many instructors know better. They let the student in on the secret of how these mysterious quantities η and δ are actually *discovered.* But in so doing the direction of the argument is reversed, and eventually they have to abandon this unorthodox discussion and recast the official proof in more-or-less the form above (or at least mention that this recasting *should* be done).

The argument in the following structured proof resembles this informal discussion but at the same time it is quite formal and rigorous. Thus the structural approach brings closer the human process and the formal-deductive one (p.179).

A structured proof

Level 1. Let $\varepsilon>0$ be given. We find (in level 2) a $\delta>0$ such that $0<|f(x)g(x)-LM|<\varepsilon$ whenever $0<|x-a|<\delta$. Thus the theorem is proved.

In the Elevator. We have to show that the expression $|f(x)g(x)-LM|$ can be made as small as we please. To this end, we try to bound it by an expression we know can be made small. Such expressions are $|f(x)-L|$, $|g(x)-M|$ and multiples of these by a constant and by each other. After some trial and error the following expression emerges:

(*) $f(x)g(x)-LM=L(g(x)-M)+M(f(x)-L)+(f(x)-L)(g(x)-M)$.

Level 2. Using the equality (*) we have:

$$|f(x)g(x)-LM| = |L(g(x)-M)+M(f(x)-L)+(f(x)-L)(g(x)-M)| \leq |L|\,|(g(x)-M)|+|M|\,|(f(x)-L)|+|(f(x)-L)|\,|(g(x)-M)|.$$

We find a $\delta>0$ (in Level 3) such that whenever $0<|x-a|<\delta$, each of the terms on the right-hand side is smaller than $\varepsilon/3$. Thus the left-hand side is smaller than ε, as required.

In the Elevator. To get $|L|\,|g(x)-M|<\varepsilon/3$, we try to make $|g(x)-M|<\varepsilon/3\,|L|$. However, there is a bug here: this only works if $L\neq 0$. One way of correcting this bug is to replace $|L|$ by $1+|L|$. The case of $|M|\,|f(x)-L|$ is similar. Finally, to get $|f(x)-L|\,|g(x)-M|<\varepsilon/3$, we make each of the factors smaller than $\sqrt{\varepsilon/3}$.

Level 3. We choose positive δ_1, δ_2, δ_3, δ_4 such that the following hold:

$|f(x)-L| < \varepsilon/3(1+|M|)$ whenever $0<|x-a|<\delta_1$;

$|g(x)-M| < \varepsilon/3(1+|L|)$ whenever $0<|x-a|<\delta_2$;

$|f(x)-L| < \sqrt{\varepsilon/3}$ whenever $0<|x-a|<\delta_3$;

$|g(x)-M| < \sqrt{\varepsilon/3}$ whenever $0<|x-a|<\delta_4$.

(Such δ_i's exist since L and M are the limits of f(x) and g(x) respectively). Now let δ be the smallest of δ_1, δ_2, δ_3, δ_4, so that if $0<|x-a|< \delta$ then $0<|x-a|< \delta_i$, i=1,2,3,4. Then whenever $0<|x-a|< \delta$, the expressions $|L||g(x)-M|$, $|M||f(x)-L|$,and $|f(x)-L||g(x)-M|$ all become smaller than $\varepsilon/3$. Thus δ satisfies the requirements of Level 2.

Remark. As seen from this example, structural proofs take longer to deliver, but (I believe) are shorter to digest. In fact, they are longer because they contain more information (namely, the structure of the proof), and it is this very information that makes them more learnable, illuminating and humane. Thus switching to structured proofs we simply agree to share with our students (or readers) more of what we know about the proof. And it is my belief that the loss in economy is more than balanced by the gain in learning.

According to Leron, the use of a structural style allows us to better communicate the ideas behind the formal proofs; namely, the main idea is given in the top level, auxiliary ideas are packaged in autonomous modules, and the interconnections between the separate ideas are made explicit through the structural diagram. Furthermore, using such an approach it is possible to be a bit more specific about what is meant by the "main idea" of a proof. The main idea often lies in the construction of a new, intermediate object, called by the author the *pivot*, to mediate between the hypotheses and the conclusion. On the contrary, in the linear approach the pivot is treated poorly (from the learner's point of view) and its potential for revealing the architecture of the proof is wasted. On the contrary, it is here where the proof most resembles the pulling of a rabbit from a hat. The pivot is usually introduced near the beginning of the proof by a bare statement of its definition, which often appears extremely bizarre and complicated. Such definitions have an intimidating, even paralyzing, effect on many students when introduced too abruptly.

Another approach studied by Blum and Kirsh (1991), investigated teaching students to understand and produce proofs using what they call a preformal presentation. In such an approach the teacher leaves out formal details while explaining the overall structure of a proof.

Another very important issue that has been deeply talked about is the role of the teacher in the process of teaching-learning-understanding proofs. Lampert, Rittenhouse and Crumbaugh (1994) described, and positively impressed, a class of fifth graders who were engaged in group discussion where the context of instruction was such that it was possible, as they put it, "for the teacher to step out of the role of validator of ideas and enter into the role of moderator of mathematical arguments."

Another teaching experiment conducted with 4th graders (M.G. Bartolini Bussi, et al., 1999), in the field of experience of gears, evidenced that, given a suitable sequence of tasks and proper teacher guidance, most of the students can produce general, abstract and conditional statements about motion in the field of experience of gears and take part in the construction of proofs as justifications inside a theory.

We could conclude with the following citation:

> The introduction of concrete referents into school mathematical activity has been debated fro years (Sierpinska, 1995). 'Realistic mathematics' (Freudenthal, 1983; Treffers, 1978) and the application of the principle of 'operative concept formation' (Bender and Schreiber, 1980) are an expression of a positive attitude. Several reasons are produced to justify the recourse to a 'real' context: pupils' motivation to learn geometry; the need to establish links between school learning and everyday learning; the conceptualization of mathematics as either 'a language to describe and interpret reality' or as 'a structure that organizes reality'. These are all pedagogical, social or philosophical reasons and each can be contrasted with different options. With this exploratory research study we hope to have taken a step ahead, illustrating the cognitive counterpart of activity with everyday concrete referents (in the case of gears) that allows early approach to theoretical thinking. (ibid, p.85)

3. THE CORE OF THE RESEARCH

3.1 The core of the research

Initially, the research was based on the idea to build a cognitive model applicable to the analysis and understanding of possible student mechanisms and difficulties related to the approach to proofs in mathematical analysis. To this extent the primary goal was to explore the creative phase of the proving process (that phase where one looks for or builds the hypothesis aimed at justifying or validating the facts proposed by the problem).

The issue of creativity in the hypothesis creation process led me to read Charles S. Peirce's works and his definition of *Abduction:*

> […] Abduction is where we find some curious circumstances, which would be explained by the supposition that it was a case of a certain rule, and thereupon adopt the supposition […] (Peirce 2.624)

Therefore, abduction is any creation hypothesis process aimed at explaining a fact. Such definition can be schematized as follows:

> F fact
> H hypothesis
> If H were true $H \rightarrow F$ therefore H is likely.

Furthermore,

> The surprising fact C is observed.
> However if A were true, C would be a matter of course.
> Hence, there is reason to suspect that A is true (CP. 5.188-189, 7.202)

C is true of the actual world and it is surprising, a kind of state of doubt we are unable to account for by using our available knowledge. C can be simply a novel phenomenon, or may conflict with background knowledge that is anomalous.

Taking into account Peirce's definition of abduction, the questions that initially guided the first steps of the research project, were:

1. When do students use abduction in proving processes?
2. If they use abduction, how do they use it? Is there a context in which they utilize it more than another?

The subsequent step was to give two different problems at two different periods of the semester to a group of students attending freshman year of an engineering degree.

Problem 1: *Let f be a function continuous from [0,1] onto [0,1]. Does this function have fixed points? (Note: C is a fixed point if f(c)= c)*

Problem 2: *Given f differentiable function in R, what can you say about the following limit?*

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$$

A first attempt of an a-priori analysis of the aforementioned problems quickly unearthed some difficulties in predicting possible student creative mechanisms. Initially the origin of such problems was not obviously apparent, but it was clear to see that the definition of abduction, as given by Peirce, was not sufficient to frame and analyze potential student creative processes.

After a deeper structural analysis of both problems I found the source of such uneasiness to be the manner in which Peirce's abduction referred to the creation of a hypothesis that could explain an observed fact.[11]

On the contrary, problem 1 contains a closed-ended question, which means a respondent can select from one or more specific categories to give the answer (in this specific case, student can choose between "Yes, the function has a fixed point", or "No, the function does not have a fixed point").

Problem 2 is an open-response task, which means a performance task[12] where students are required to generate an answer rather than select it from among several

[11] In an abductive process a "starting fact" is always considered and it is always true.

[12] A performance task is an exercise that is goal directed. The exercise is developed to elicit students' application of a wide range of skills and knowledge to solve a complex problem.

possibilities, but where there is a single correct response (definition taken from NCREL: North Central Regional Educational Laboratory).

In both cases the reader is confronted by a problem with a direct question, which means the solver not only has to find hypotheses justifying a fact, but also has to look for a fact to be justified. In conclusion, among the problems given to the students, there is not a fact already observed and definitely true. This particularity generates the need to analyze the abductive processes under a new light, in the sense that the nature of the fact and the connections between hypothesis and fact have to be considered in a different way than the manner proposed by a standard abductive process.

To clarify what I mean about this difference, let us take into consideration the following example given by Peirce:

> For example, fossils are found; say, remain like those of fishes, but far in the interior of the country. To explain the phenomenon, we suppose the sea once washed over this land. This is abduction. (Peirce, 2.624)

In this case the *truthfulness* of the fact is independent to the *truthfulness* of the hypothesis built to explain the fact; if the hypothesis, at a certain point, turns out to be false, this will not change the status of the fact, namely, the fossils would still remain "far in the interior of the country."

Therefore, in the problems considered in the experimentation, both hypothesis and fact may take the aspect of **conjecture**, which Webster's 1913 Dictionary defines as, *"an opinion or judgment, formed on defective or presumptive evidence; probable inference; surmise; guess; suspicion.*

Having arrived at this point it is necessary to clarify which meanings of the words *fact and hypothesis* will be adopted in this work.

In terms of the word 'hypothesis', Aristotle has already used this word meaning 'hypothesis of a theorem,' but Archimedes (in The Arenaria) tested a 'hypothesis' as related to physic reality, implicitly changing the sense of the word with respect to Aristotle (Boero et al., 1995). Today, the word 'hypothesis' covers a wide range of meanings. For example, in Collins Dictionary: a hypothesis is an idea that is suggested as a possible explanation for a particular situation or condition, but which has not yet been proved to be correct. Whereas, in Webster's Dictionary we find: (1) a supposition; a proposition or principle which is supposed or taken for granted, in order to draw a conclusion or inference for proof of the point in question; something not proved, but assumed for the purpose of argument; (2) a system or theory imagined or assumed to account for what is not understood; (3) the antecedent clause of a conditional statement.

Henceforward, the word '*fact*' will be defined as: *referring to something as a 'fact' means to think it is true or correct.* Whereas '*hypothesis*' will stand for: *an idea that is suggested as a possible explanation for a particular situation or condition.* While, *hypothesis,* in the Aristotelian sense (i.e.: hypothesis of a theorem), will be substituted by the term *'given.'* Such a choice is motivated by the interest of the research that deals with the creative aspects of a cognitive process and not, for example, with the formal rearrangements of a proof.

Let us go back to the terms *'hypothesis'* and *'fact'* related to the aforementioned problems. As stated, both may take the aspect of conjecture; the former is a conjecture with the role of hypothesis meant as possible explanation; the latter is a *'conjectured fact'* (in the sense that it could reveal itself to be untrue) in terms of the role of final answer to the problem, or answer to certain steps of the solving process. This kind of fact will be indicated with 'c-fact' to distinguish it from the standard fact (as defined by Collins' Dictionary).

The tenet of abduction has also been confronted by Cifarelli part of whose research is concerned with the relationships between abductive approaches and problem-solving strategies. The purpose of his work is to clarify the processes by which learners construct new knowledge in mathematical problem solving situations, with particular emphasis on instances where the learner's emerging abductions or hypotheses help to facilitate novel solution activity (Cifarelli, 1999. The basic idea is that an abductive inference may serve to organize, re-organize, and transform a problem solver's actions.

The following example given by Cifarelli (in "*Abduction, Generalization, and Abstraction in Mathematical Problem Solving,*" 1998) may highlight the core of his work:

> Marie is a student who was given a set of algebra word problems, designed by Yackel (1984) to induce problematic situations.
> Marie had to solve the first problem involving the depths of two lakes, and then she was asked to solve eight follow-up tasks, each a variation of the original problem. The problems were designed in such a way to have a range of similar problem solving situations and hence develop ideas about "problem sameness" in the course of her on-going activity.
> The third problem had insufficient information; initially, Marie guided by the sameness of the problem tried to solve it in the same way she had solved the previous two; very soon she realized that it was not possible and that became for her a novel situation. The abduction took place at this point, namely Marie needed to find an explanation of her failure.
> <<…The same way (she smiles, then displays a facial expression suggesting sudden puzzlement) impossible!! It strikes me suddenly that there might not be enough information to solve this problem (she re-reads and reflects on her work) I suspect I'm going to need to know the height of one of these things (solver points to both containers in her diagram). I don't know though, so I am going to go over here all the way through>>

Applying Peirce's logic structure, Marie's abductive process would be translated in the following way:

F: impossible (namely, the failure of the solving strategy Marie had thought to use)

H: there isn't enough information to solve the problem

If H were true $H \rightarrow F$ therefore H is plausible

Therefore, the fact is represented by *a failure* and the abduction is the search of an explaining hypothesis to such a failure.

Cifarelli's analysis of Marie's process is as follows:

> Marie's anticipation that "the same way" would not work was followed by her *abduction* that the problem did not contain enough information, later refined to the hypothesis that she needed more information about the relative heights of the unknowns. While the hypothesis contained elements of uncertainty, it helped organize and structure her subsequent solution activity, whereupon she explored and tested its plausibility as an explanatory device. (p.7).

Cifarelli's attention is focused on the abductive inference as a tool to enhance the search for further strategies when the application of a previous solution does not work.

The hypothesis of the absence of enough information leads Marie to go through the problem again to verify the plausibility of her hypothesis, and then to construct the necessary data to solve the problem. Therefore, the researcher is not interested in the “typological aspect” of abduction, but in the role such a process plays on the problem-solving activities.

Returning to the analysis of the “typology” of abduction, I am intrigued by Cifarelli’s extension of the concept of fact: in Peirce’s abduction the fact is a tangible observation: the fossils far in the interior of the country, the white beans on the table, and the documents talking about Napoleon; according to Cifarelli the fact may also be represented by something that happens (e.g., the failure of a strategy). This new point of view gives me the impetus to reflect on a new interpretation of the typology of abduction, where the fact is also represented by a strategy / procedure or regularity.

Recapitulating, Peirce’s definition is insufficient according to my research into the analysis of the cognitive creative processes, and it leads me to consider the case where both fact and hypothesis are conjectures. Furthermore, Cifarelli suggests to me the idea of looking at the typology of “ conjectured-fact” as a procedure.

Hence, the situations we can meet are:

1. The subject experiences a given fact (it is already true) and looks for a hypothesis that may explain the fact (Peircean situation)
2. The subject gives <u>an answer</u> (fact or conjectured-fact) and looks for a hypothesis that may legitimate or explain the answer or fact.
3. The subject <u>gives an answer</u> (a fact or conjectured-fact) and looks for a <u>strategy</u> that may legitimate or explain the answer or fact.
4. The subject <u>gives an answer</u> (a fact or conjectured-fact) consisting of an already known strategy applied by him to a novel situation, and looks for <u>tools</u> that may legitimate such an adaptation.

As a consequence of these new considerations about abductive processes, the research questions can be modified as follows:

1. Are the definitions of abduction, already given, sufficient to describe creative processes of an abductive nature? Or, is a broader definition of abductive process needed to describe some creative students' processes in mathematics proving? If so, what is that definition?
2. Is one's certainty about the truth of an assumption an indication for an initiation of abductive reasoning in her or his process? Namely, how much is important the level of confidence of the built answer to guide an abductive approach?
3. Is there continuity between the cognitive "tool" one uses to build a conjecture and the means one uses to establish its validity?
4. Which elements convey an abductive process? In particular, does transformational reasoning facilitate an abductive process?

At this point it is necessary to ask the question, " What links these research questions with Peirce's work? The common denominator is the philosophic spirit on which both works are based. The core idea is the intention to show that the creative process owns some components, and to separate this process from the belief that it is not possible to talk about it because it is something indefinable and only comparable to a "flash of genius." This is the philosophical foundation of Peirce's work, a man who *"…struggled over more than fifty years to lay bare the logic by which we get new ideas"* (Fann, 1970). Peirce wanted to legitimate the fact that abduction is a kind of reasoning along with deduction and induction, and he was willing to show that *"…reasoning towards a hypothesis is of a different kind than reasoning from a hypothesis"* (ibid.), in contraposition with other philosophers like Popper who claimed that *"…the initial stage, the act of conceiving or inventing a theory, seems to me neither to call for logical analysis nor to be susceptible of it"* (Popper, 1959). Many philosophers regard the discovery of new ideas as mere guesswork, chance, insight, hunch or some mental jump of the scientist that is only open to historical, psychological, or sociological investigation. The attempt of this research is to build a cognitive model that will help to recognize creative processes.

3.2 The Abductive System

According to the initial difficulties of analyzing the problems using only Peirce's definition of abduction, and the new considerations made about tasks requiring not only the construction of a hypothesis but also of the answer, I have constructed new definitions and tools which have been employed in the analysis of the protocols.

I define the **Abductive System** as being a set whose elements are: *facts, conjectures, statements,* and *actions*: **AS** = {facts, conjectures, statements, actions}.

For *fact* I adopt the definitions of Collins' Dictionary: (1) *referring to something as a* ***fact*** *means to think it is true or correct;* (2) ***facts*** *are pieces of information that can be discovered.*

For *conjectures* I adopt the definition given by the Webster's dictionary: ***conjecture*** *is an opinion or judgment, formed on defective or presumptive evidence; probable inference; surmise; guess; suspicion.*

The *conjectures* assume a double role of:

1. *Hypothesis*; an idea that is suggested as a possible explanation for a particular situation or condition.
2. *C-Fact (conjectured-fact)*; final answer to the problem, or answer to certain steps of the solving process.

Facts and Conjectures are expressed by *statements* divided into the three following categories:

1. Stable statements
2. Unstable statements
3. Abductive statements

A *stable statement* is a proposition whose truthfulness and reliability are guaranteed, according to the individual, by the tools used to build or consider the fact or conjecture described by the proposition itself. Namely, the truthfulness depends directly on the tools employed in the construction phase (E.g. a "visually-based" fact: the validity of the proposition describing the phenomenon is justified by a visual perception).

An *unstable statement* is a proposition whose truthfulness and reliability are not guaranteed, according to the individual, by the tools used to build or consider the conjecture described by the proposition itself. Namely, the tools used in the creation phase are not sufficient for the solver to consider the conjecture described by the proposition as being definitively true. The consequence of this is the search of a hypothesis and or an argumentation that might validate the aforementioned statement.

An *abductive statement* is a proposition describing a hypothesis built in order to corroborate or to explain a conjecture. The abductive statements too, may also be divided into stable and unstable abductive statements. The former, according to the solver, state hypotheses that do not need further proof; the latter require a proof to be validated, that means a process that brings back and forward.

An abductive statement may present different structures:

1. It describes a hypothesis to justify a conjecture.
2. It describes a procedure to justify a conjecture.
3. It describes tools to justify a procedure.

It is important to clarify that the definitions of *stable*[13] and *unstable statement* are student-centered, namely, the condition of stable and unstable is related to the subject: what can be stable for one student may represent an unstable statement for another student and vice versa; not only that, but the same subject may believe stable a particular statement at a certain point of their scholastic career, and this may become unstable later on when their base cultural knowledge of structured mathematical knowledge increases (e.g.; she or he learns new mathematical systems; new axioms and theorems). Furthermore, a stable statement may become unstable, inside a similar problem solving

[13] The concepts of stable and unstable are related, moreover, to the mathematical context. In Euclidean Geometry if a statement is stable, the problem will be only to find the tools to prove it. Namely, in Euclidean Geometry it is enough to find few variations of "targeted" drawings to guarantee the stability of a statement. In Arithmetic the problem is more complex; it is sufficient to think of Goldbach's conjecture. Goldabach's original conjecture (sometimes called the "ternary" Goldbach conjecture), written in 1742 in a letter to Euler, states "at least it seems that every number that is greater than 2 is the sum of three primes". Note that here Goldbach considered the number 1 to be prime, a convention that is no longer followed. As re-expressed by Euler, an equivalent form of this conjecture (called the "strong" or "binary" Goldbach conjecture) asserts that all positive even integers ≥ 4 can be expressed as the sum of two primes. Not only a proof has not been found yet, but also, even though many millions of even numbers have satisfied such property, we are still not sure of its validity.

process, not because the student is convinced of that, but for a "cultural contract"; namely, the student may recall their scholastic experience and remember that a statement is considered stable if it is justified inside a precise mathematical system supported by axioms, and theorems; thus they will analyze the tools employed for verification if they satisfy such conditions. Another situation leading the student to reconsider a statement from stable to unstable is the "didactical contract"; the subject might believe the visual evidence to be sufficient in order to justify a conjecture, but the intervention of the teacher could underline its insufficiency and therefore the students would find themselves looking for new tools. Furthermore, the same statement may transform from unstable to stable inside a similar process because the subject follows the mathematician's path: they starts *browsing* just to look for any idea in order to become sufficiently convinced of the truth of their observation, then they turn to the *formal-theoretical world* in order to give to their idea a character of reliability for all the community (Thurston, 1994).

The following example, taken from Harel's *Proof Schemes* work, seeks to clarify part of this tension:

> […] Further, a person can be certain about the truth of an observation in one situation, but seek additional or different evidence for the same observation in another situation. For example, long before students learn geometry in school, they are convinced, based on personal experience and intuition, that the shortest way to get from one point to another is through the line segment connecting two points. Later, as participants in an Euclidean geometry class, an instantiation of this observation - stated in the theorem "The sum of the lengths of two sides of a triangle is greater than the length of the third side" – may become a conjecture for the students until they find evidence that would be accepted by their class community or their teacher. The kinds of evidence the students may look for are based on whatever conventions are accepted in their class as evidence for a geometric argument. These conventions may differ from one class to another; for example, what might be accepted as evidence in a standard high school Euclidean geometry class is likely to be insufficient evidence for a college class studying axiomatic geometry. (p. 243)

Behind any statement there is an action. *Actions* are divided into <u>*phenomenic* actions</u> and <u>*abductive* actions</u>. A *phenomenic action* represents the creation, or the "taking into consideration" of a fact or a c-fact: such a process may use any kind of tools; for example, visual analogies evoking already observed facts, a simple guess, or a feeling, “that it could be in that way”; a phenomenic action may be guided, for example, by a didactical contract or by a transformational reasoning (Harel, 1998).

An *abductive action* represents the creation, or the “taking into account” a justifying hypothesis or a cause; like the phenomenic action, they may be conveyed by a process of interiorization (Harel, 1998), by transformational reasoning (ibid) and so on. The abductive actions may look for:

1. A hypothesis, to legitimate the previous met or built conjecture
2. A procedure, to legitimate or justify the previous built conjecture
3. Tools to legitimate the adaptation of an already known strategy to a novel situation.

After a broad description, the Abductive System could be schematized in the following way: *conjectures* and *facts* are ‘act of reasoning’ (Boero, 1995) generated by phenomenic or abductive actions, and expressed by ‘act of speech’ (ibid) which are the statements. The adjectives *stable*, *unstable*, and *abductive* are not related to the words of the statements but to the acts of reasoning of which they are the expression. Hence, the only tangible thing is the act of speech, but from there we may go back to a judgment concerning the act of reasoning thanks to the adjectives given to the statement.

Finally, for two different subjects the same statement may be stable or unstable. Therefore, two persons may achieve the same act of reasoning and judge it by a different method.

The following chart shows the structure of the Abductive System

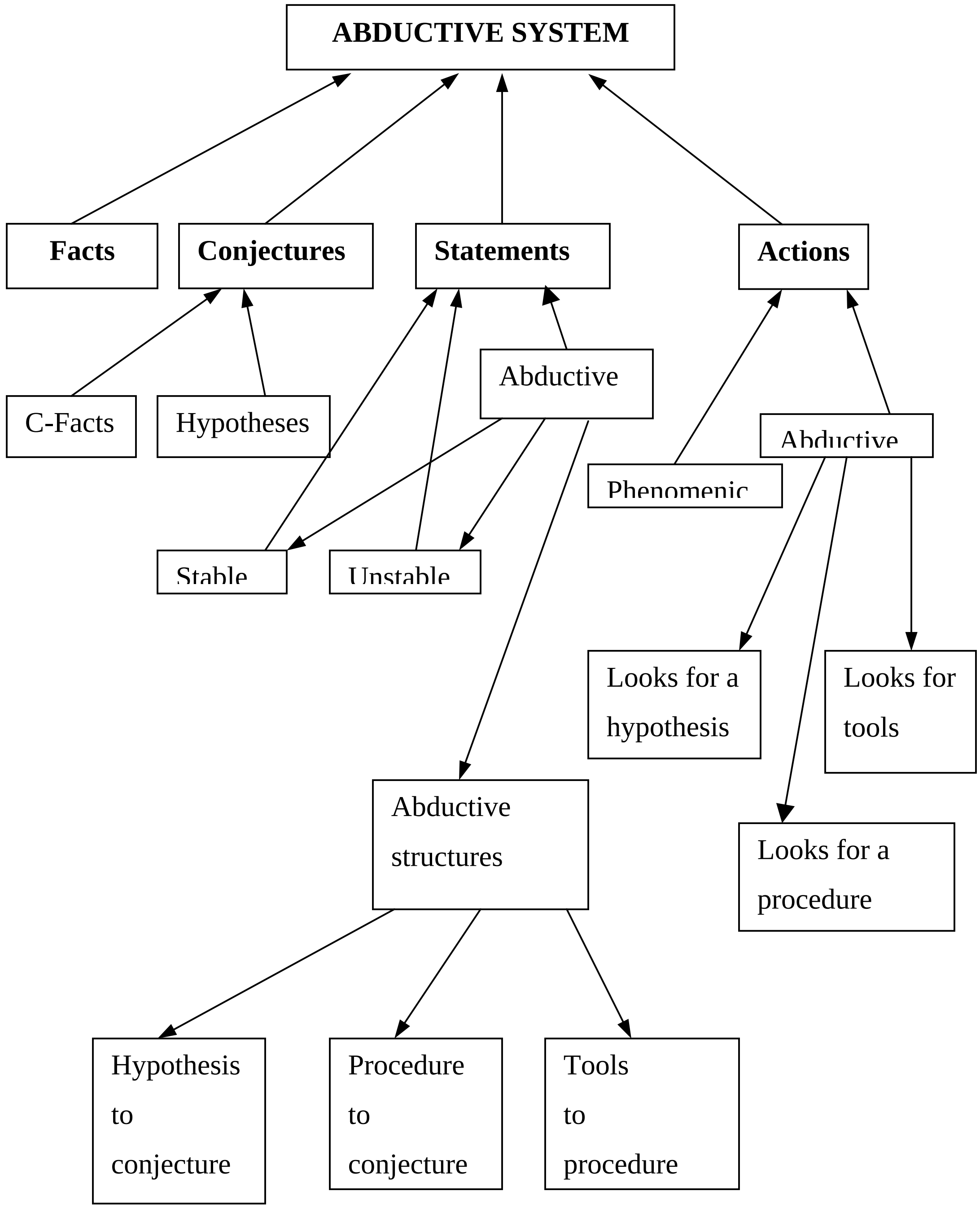


Figure 6: The Abductive System

4. METHODOLOGY

4.1 Site and Participants

The study is a *basic research*[14] and its purpose is to build a model to identify and account for possible cognitive processes students implement when they perform conjectures and proofs in Calculus, specifically a cognitive model that will help to recognize creative processes.

The data has been collected at the University of Industrial Engineering and Management of Genova (Italy) during the academic year 2001-2002, and the participants are freshmen enrolled in required calculus classes for engineers. The courses cover differentiation and integration of one-variable functions as well as differential equations. The student participants are 18 or 19 years old. There are two main reasons for choosing to work with this population: 1) My working experience is with students of this age; 2) The approach of the university frequently revealed a very delicate and difficult issue, since the "cultural and didactical reality" the students come in contact with at the university is markedly different from their experiences in high school. This gap, in many cases, seems to be critical for the mathematical development of these students. The university approach demands more autonomy in facing mathematical problems. This approach asks students to participate in autonomous work in the creation of hypotheses, conjectures and implement a sense of critique in evaluating their own actions in the problem solving process; such a request seems to cause to the students several important problems, suggesting their creative abilities had been lost during their scholastic career.

At the beginning of the Calculus course the teacher introduced me to the students as a Teacher Assistant, working once a week with them in class for a session of three

[14] The purpose of basic research is knowledge for the sake of knowledge. The basic researcher's purpose is to understand and explain. The most prestigious contribution to knowledge takes the form of a *theory* that explains the phenomenon under investigation. (Patton, 1990)

hours, during which the students would solve problems proposed by me, and they would be able to discuss possible problems raised by them. During the week, the students would be able to come to my office for further explanations about topics discussed in class, or about exercises solved autonomously.

The students' participation at the lessons was not mandatory, and there was not any relation between their participation and the result of the final exam, since the teacher would never know who followed the lessons and who did not.

Having established my main role (namely, the one described above), I later asked the students if someone was interested in taking part in a research project, which was related to my doctoral thesis. I clarified that such participation would not be mandatory, and that there would not be any relationship between their consent and the results of their final exams. I explained that the purpose of my study would be to look for possible creative processes during the problem-solving phase; and to this extent I would give the participants in the project some tasks to solve, and they would be videotaped, and that I would participate in some lessons given by the teacher in order to gather field notes.

The choice of the classroom participants (about one hundred students) was completely left to this group of students and was therefore totally random; my only concern being that the sample would be heterogeneous from the point of view of both culture and ability; but this could be monitored since I was constantly in contact with the students.

4.2 Data Collection

The data was collected through the following sources:

a) <u>One questionnaire</u>, distributed to all of the students (about one hundred) of the classroom; the questionnaire was anonymous, and composed of the following questions:

1. CHECK THE FORMS OF REASONING YOU KNOW
   - ❑ Induction
   - ❑ Deduction
   - ❑ Others. Which ones?

2. AS A STUDENT, DO YOU CONSIDER THE STUDY OF PROOFS TO BE NECESSARY?
    - ❑ Yes. Why?
    - ❑ No. Why?
    - ❑ Sometimes. When?
3. WHICH KIND OF RELATIONSHIP LIES BETWEEN HYPOTHESIS AND THESIS IN THE CONSTRUCTION OF THE STATEMENT OF A THEOREM?
    - ❑ The hypothesis always comes before the thesis. Why?
    - ❑ The thesis always comes before the hypothesis. Why?
    - ❑ Depends (Justify it)
4. FOR EACH THEOREM DO YOU THINK THAT THERE EXISTS ONLY ONE CORRECT PROOF?
    - ❑ Yes. Why?
    - ❑ No. Why?
5. THE CONSTRUCTION OF A PROOF HAS TO FOLLOW A FIXED PATTERN. CREATIVITY CANNOT FIND ROOM IN THE CONSTRUCTION OF PROOFS.
    - ❑ True. Why?
    - ❑ False. Why?

Note: for the following question it is possible to choose more than one answer

6. A PROOF IN CALCULUS HAS THE FOLLOWING ROLE(S)
    - ❑ Convince someone of the validity of a statement
    - ❑ Explain why a statement is valid
    - ❑ Establish the validity of a statement
    - ❑ Other (specify)

The purpose of the questionnaire is to investigate what kind of “culture of proof” students own; and what conceptions and misconceptions they have about this issue, in order to understand the kind of cultural background owned the potential participants at the research project.

What I am interested in is the idea students have about the construction of proof, its use and role also from a didactical point of view, and how their “scholastic experience" may have influenced and changed the way they think of proof and how they think of themselves in relationship with the construction of a proof.

The first question is just a survey tool to check which forms of reasoning students are familiar with, and whether they define or recognize forms of reasoning other than the inductive and deductive ones.

The second question is designed to investigate what kind of "mental attitude" students approach a proof with, if they tackle the construction of a proof just because they are told to do so by the teacher, or if there is a sort of curiosity and a conviction about its necessity. The "why" question is designed to examine what kind of influence school could have had in students' opinion about such an issue.

Question number 3 attempts to discover and analyze students' conceptions about the structure of a proof. Very often students are involved in dealing with "ready made" proofs; their first experience with such proofs usually represented by the presentation of the statement of a theorem followed by a well structured proof, meant as a chain of deductive steps, one following the previous one, supported by axioms or previously proved properties or theorems. Very often, mathematician's cognitive processes, employed to generate such a proof, are an alien topic for the students themselves. Unfortunately, this means that students very seldom have the opportunity to deal with a "proof in progress." On the contrary, they usually have experience with the kind of didactical contract that sees the teacher as the only source of truth, and the one who simply transfers some pre-constructed knowledge to the class.

In a similar manner, questions 4 and 5 aim to understand the ideas, regarding proofs, students have constructed during their scholastic careers. The question also seeks to determine if they think it is possible for any theorem to have just one correct proof, or if they do not relate creativity and personal initiative with the process of constructing a proof, because over time they only experience the final product of the proof. If so, we may interpret their difficulty in their approach to the proving process and their reluctance to tackle an open problem because they just wait for somebody tells them how to proceed. The final question is critical because it is fundamental in attempting to understand which role students give to a proof, because it is this idea that leads their predisposition toward the construction of the proof.

Briefly, from the analysis of the questionnaire most of the students think of proofs as a tool to better understand theorems, their meaning, and the reasoning involved into the process of proving. The remaining part is mainly concerned with the idea that proofs are necessary because they validate the problem and convince of its validity, or as a tool useful to solve problems, to create mental schemes to be used in problem-solving, furthermore they explain the why of a fact, and finally they make a context clearer, and easier to be remembered. Furthermore, the totality of the students agrees with the fact that there may exist more than one correct proof for the same theorem. Finally, creativity seems to be an important component for the construction of a proof. (A complete analysis of the questionnaire can be found in Appendix A).

b) Two different exercises given, at two different periods of the semester, to the participants in the project (twenty students took part in the project). In the problem solving phase the participants were asked to work in pairs (leaving to them the decision about whom to work with); the choice of making them work in pairs was motivated by the conviction that the necessity of "thinking aloud" to communicate their own ideas gives the opportunity to bring to light guessing processes, creations of conjectures and their confutations, namely those creative processes which in great part remain "inside the mind" of the individual when one works alone, and very often only the final product is communicated to the others (Thurston, 1994; Lakatos, 1976; Harel, 1990; et al.).

It is important to note that the participants were not asked to produce any particular "structured" solution; my aim being to leave the students completely free to decide their solution process and to autonomously evaluate the acceptability of their solution for the learning community.

Problem 1**:** *Let f be a function continuous from [0,1] onto [0,1]. Does this function have fixed points? (Note: C is a fixed point if f(c)= c)*

At the time the students were given this problem they had already been exposed, in the curriculum, to the theory of the continuous functions, with related theorems, but they had not previously seen the definition of fixed point. This problem contains a close-ended question, which means respondent can select from one or more specific categories to give the answer (in this specific case, student can choose between "Yes, the function has a fixed point" or, "No, the function does not have a fixed point").

Problem 2: *Given f differentiable function in R, what can you say about the following limit?*

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}.$$

At the time this exercise was proposed, the students have been exposed to the definition of differentiable function through the limit of the difference quotient. Problem 2 is an open-response task, which means a performance task[15] where students are required to generate an answer rather than select it from amongst several possibilities, but there is a single correct response (definition taken from NCREL: North Central Regional Educational Laboratory).

In both cases the reader is confronted by a problem with a direct question, which means the solver not only has to find hypotheses justifying a fact but also identify a fact to be justified.

The aim of these exercises is to confront the students with the necessity to produce conjectures, to prove their validity or refute them, namely to use their own creative processes in order to produce facts, conjectures and hypotheses.

Throughout the process of both problems the students have been videotaped.

c) Videotape of a lecture given by the teacher. The aim of the collection of this data is to study the behavior of the teacher during the didactical transposition, and observe the relationship between the "cognitive attitude" of the teacher and that of his students.

[15] A performance task is an exercise that is goal directed. The exercise is developed to elicit students' application of a wide range of skills and knowledge to solve a complex problem.

4.3 Data Analysis

The analysis of the protocols is based on the analysis of the text (which has been transcribed verbatim from the videotape) with the aim of looking for possible structures in the dialogue indicating creative processes, meant as the processes of creation of facts, hypotheses and conjectures.

From the analysis of the dialogue I want to find which kinds of reasoning enhance a creative attitude. Besides the analysis of the text, I want to analyze what the students have produced in their protocols, in order to look for possible relationships among the various languages: from the graphic language, iconic and algebraic, and the process of creation of hypotheses, conjectures and facts.

Therefore, I want to understand how students make sense of mathematical symbols, in which ways they interact with icons and graphs in order to create hypotheses, conjectures and facts.

The analysis of the protocols is divided into two phases. The first phase shows a comprehensive description of students' behaviors in tackling the problem; in the second phase the creative processes are detected and interpreted through the elements of the abductive system.

The videotape is a tool in the triangulation of the data; it gives the opportunity of going over any dialogue students have engaged in during the problem solving process. In the same way, the analysis of the transcript of the lecture given by the teacher is aimed at examining the structure of the teacher's dialogue, indicating creative processes, and to compare these with the attitudes observed in the students.

My theoretical framework is based on the notion of *Symbolic Interactionism*. Jacob (1987) states that the focus of Symbolic Interactionism is to understand the processes by which points of view develop. And such a tradition provides models for studying how individuals interpret objects, events, and can be utilized for studying how this process of interpretation leads to certain behavior in specific situations.

Concerning the analysis of the data: *Content Analysis*[16] has been adopted, in the sense that Content Analysis is the process of identifying, codifying and categorizing the primary patterns in the data (Patton, 1990).

[16] *Content analysis* is the process of identifying, codifying, and categorizing the primary patterns in the data. This means analyzing the content of interviews and observations. (Patton, 1990)

5. ANALYSIS OF THE DATA

5.1 Analysis of the protocols

The analysis of the protocols is based on the analysis of the text (which has been transcribed verbatim from the videotape) with the aim of looking for possible structures in the dialogue indicating the creative processes, those of the creation of facts, hypotheses and conjectures. Besides the analysis of the text, I want to analyze what the students have produced in their protocols, in order to look for possible relationships among the various languages: from the graphic language (iconic and algebraic), and the process of creation of hypotheses, conjectures and facts. The analysis of the protocols is divided into two phases:the first phase shows a comprehensive description of students' behaviors in tackling the problem; the second phase illustrates how the creative processes are detected and interpreted through the elements of the abductive system.

Tables divided into two columns represent the structure of the second phase of the analysis; the left column is used to write the excerpts considered relevant to the creative processes; while the right column has been used to write the interpretation of the excerpts through the tools of the abductive system; furthermore the vertical arrows linking one excerpt to another describe the possible cognitive movement leading from one statement to another one. Let us revisit the text of the two problems:

<u>Problem 1</u>**:** *Let f be a function continuous from [0,1] onto [0,1]. Does this function have fixed points? (Note: C is a fixed point if f(c)= c)*

<u>Problem 2</u>: *Given f differentiable function in **R**, what can you say about the following limit?*

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}.$$

5.1.1 Marco and Matteo (fixed point problem)

The first step is represented by the statement: *"the function probably has a fixed point"*. This consideration seems to be generated by a mechanism of didactical contract, namely, if the problem asks such a question…it follows that the answer is affirmative….

The aforementioned <u>act of reasoning</u> is expressed by an unstable statement, in the sense that Matteo and Marco do not consider the request of the existence of a fixed point, a sufficient reason to legitimate what they have claimed. Hence we can define it a <u>c-fact</u>, because it plays the role of final answer to the problem but the subject is not sure of its truthfulness.

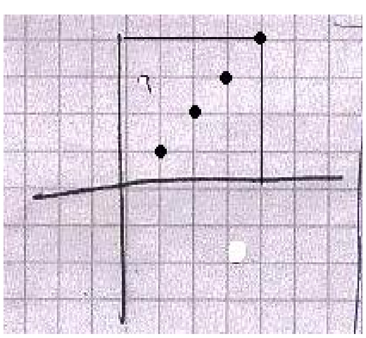
At this stage the attention shifts backwards: before looking for a hypothesis justifying the presence of a fixed point, Marco and Matteo try to understand which the fixed points are and how they can be found; the core of the problem now becomes *to identify and explicate the properties of the set of the fixed points;* they need to create a *cultural background*, meaning a theory supporting the creation of the hypothesis.

R7: Matteo: *How can we find this fixed point?*

This is the phase of the construction of a theory aimed at the creation of the hypothesis; namely, "if we can understand how the fixed points are made and how we can find them, then we will be able to create a hypothesis that might justify the presence of fixed points."

They try to understand which ones are the fixed points, and they say:

R8: *a fixed point is here, another one is here…* (see Figure 7) and they arrive at the conclusion that the fixed points lie on the bisector line.



Then

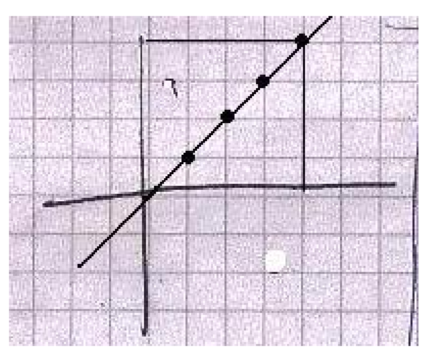


Figure 7: Representation of the fixed points

The construction phase of a possible theory is characterized by a graphic exploration. The graphic aid comes into this: Marco and Matteo, led by the squares on the paper, start identifying the fixed points with ones of the vertexes of the squares, because they satisfy the condition to have the same coordinates, and from the visualization in the discrete they go to the continuous, hypothesizing that if it is valid on the visible vertexes, it will be valid for all the "sub-squares" which is made by. Marco and Matteo have the following definition of fixed point: (c, f(c)) with f(c) = c; therefore *c* is the "x" and *f(c)* is the "y". The subsequent step is represented by their statement that the fixed points are the ones that have "the x equals y" and the y represent it graphically as vertexes of the square of the paper. The idea that the point has the same coordinate allows Marco and Matteo to sign them on the vertexes of the square of the paper. Therefore, the idea is translated in sign, such a sign potentially allows a new step, it suggests visually the passage from discrete to continuous…they probably realize, thanks to a visual factor, that between the square represented by the first square of the paper and the second one there are other infinite squares whose vertex will represent a fixed point. Therefore, they draw the line connecting these points; always working graphically they realize that what they have just drawn is the bisector line of the I and III orthant and therefore there is a shift to the interpretation of the fixed point represented by the passage from f(c)= c to y = x (again the sign is source of thought, a dynamic that goes from outside to inside). There is an identification of the set of the fixed points with the bisector line of the I and III orthant. Therefore in the passage from the discrete to the continuous the graph becomes a source…meant as a new source of thought.

We could schematize these steps as follows:

- The vertexes of the visible squares of the paper sheet represent fixed points
- Among the visible squares there are infinite other squares:

Then *Consideration of a fact corroborated by a sub-intended hypothesis, namely "the space between the squares is not empty"; hypothesis that seems not to need further proofs of its validity.*

- There are infinite vertexes representing fixed points, and a line can link these points

Then

- Visually I realize that such line is the bisector line of the I and III orthant.

At this point Marco and Matteo, in their theoretical background, own the following notions:

1. The set of the fixed points is the bisector line (*built by the student*)
2. Function f, continuous in [0,1] onto [0,1] (given *of the problem*)
3. Continuous function means that there are no gaps in the interval [0,1] (*students' built conception*)

As we can understand from the following excerpt:

R10**: Matteo:** *I would say yes…I would say that the fixed points are on…y = x…and if our function must assume all the value of the image in such a way if it is continuous it must go through this line…there will be a point for sure…*

The graphic representation leads to state that there is an intersection with the bisector line.

The situation so far is schematized as follows:

1st STEP: the function has fixed points (*if the problem asks it…*didactical contract)

2nd STEP: the set of the fixed points is the bisector line (*built by the student*)

3rd STEP: the function intersects the bisector line.

The act of reasoning expressed by the statement "the function intersects the bisector line", takes a complex aspect. It comes out as the consequence of a graphic representation of the bisector line and several continuous functions, standing for the answer to a certain phase of the solving process; subsequently the statement is re-interpreted and the c-fact becomes a hypothesis, meant as possible explanation of the initial conjectured-fact "the function probably has fixed points."

The act of speech becomes an unstable abductive statement, because Matteo doesn't think of the visual impact as a sufficiently strong justification to guarantee that all the continuous functions in [0,1] intersect the bisector line in the interval.

Before proceeding with the analysis of the protocol the following observation is needed: in the first phase it has been said that the act of reasoning represents a c-fact, relating its "instability" to that one considered when the act of reasoning takes the role of hypothesis, nevertheless, we do not have to ignore the hypothesis that if the act of reasoning had been stopped at the first step, the visual impact could have been enough for Matteo and Marco, and then the act of reasoning would have been expressed by a stable statement.

The subsequent step is to prove the validity of the hypothesis "gr(f) $\cap$ b $\neq \emptyset$". In the proving phase Matteo and Marco use the proof by contradiction; such a strategy is probably conveyed by the fact that in the lessons immediately prior to the experimentation, the students had met this kind of proving approach, and therefore they try to use it in the current situation. Furthermore, a proof by contradiction leads the two students to work with the existential quantifier $\exists$, instead of the universal quantifier $\forall$, which may represent an easier argument. Let us take into consideration the following excerpt:

Matteo tries to explain to Marco:

R16: **M:** *we have to prove that f (x) intersected with y = x is not empty, different to the empty set. We have to prove that it is possible to go from here to there without intersecting the bisector line, but if a > b taking a as the point where x = 0 and that lies on the upper side of the bisector line, b the point where y = 1 and b lies on the lower side of the bisector line there must be a point between the two where the x = y…there must be for sure and I can do the same thing changing the position of the two points respectively…or collocating them at the same height…I have to write it down in formal way*…(conception of proof as formal proof) (see Figure 8).

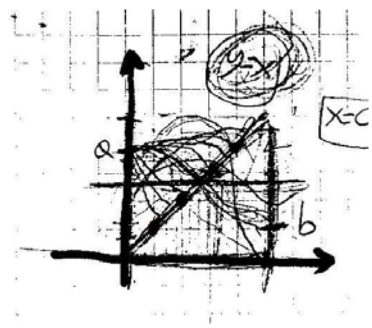


Figure 8: Matteo's graphic aid

They try to build an acceptable standard proof and at this point I give them the formal definition of continuity with ε and δ. They try for a while but then they go back to the proof by contradiction.

Now there is a shift that moves the individual argumentation process (a dialogue inside the self) towards an audience. I think it is still an "ascertaining" phase and not "persuading." The communication toward the other person seems to be a new tool to make argumentative inquiry; namely, Matteo is trying to explain to Marco his own point of view, in doing so I think he is trying:

- To shed more light on his own argumentative process
- To find in his interlocutor assistance to overcome an impasse in which he, Matteo, seems to be.

Therefore, Matteo reformulates the fact previously stated: "we have to prove that f (x) intersected with y = x is not empty".

On the graph he visualizes the two points (0,a) with $0 < a \leq 1$ and (1,b) with $0 \leq b < 1$ underlining that the first point lies on the upper side of the bisector line and the second point on the lower side (important, because this implies that the bisector line "interferes" with the graph of f(x)).

R17: **Matteo**: *by contradiction we take 'a' that is greater and ≠ 0 and 'b' minor, now we say by absurd it doesn't go to, at this point 'a' will take in this point here any point in the middle and that a ≠ y, therefore a point in which y > x always because in the first instance we said that it was greater therefore y must be greater than x and in this other* *little point here and here and here it will always be greater strictly greater we arrive here where it must be greater than x, at this point we have to take all these points here; its value in 1 cannot be less than 1, equal 1 or more than 1 because it must stay in this interval here, therefore it is absurd.* (Figure 9)

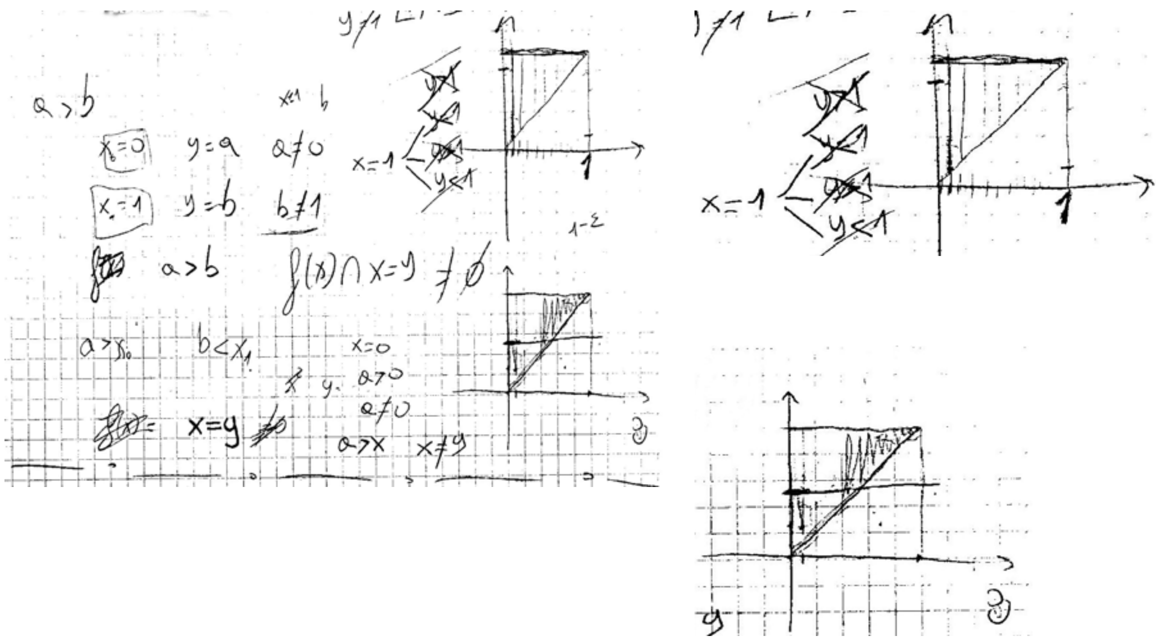
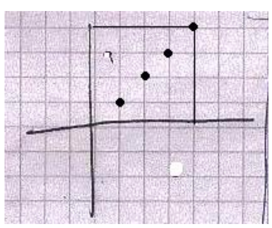
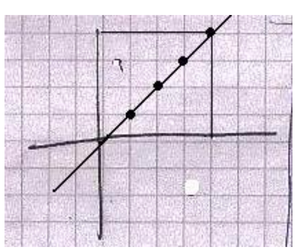


Figure 9: Matteo and Marco's graphic attempts

In the proving phase it is possible to identify a further creative movement. Adopting the approach of the proof by contradiction, the hypothesis to be proved becomes: gr(f) $\cap$ b = $\emptyset$. This occurs through the graphic representation and the use of transformational reasoning (Harel), meant as the ability of reasoning dynamically on the graph of the function, and of anticipating the possible results of such graphic-dynamic exploration. Matteo arrives at a new act of reasoning expressed by an <u>abductive statement</u>: *the graph of the function belongs totally to the upper triangle.* The aforementioned hypothesis would explain why gr(f) $\cap$ b = $\emptyset$. At this stage they use a transformational reasoning to prove that having arrived at x=1 none of the following options would be acceptable, y=1, y<1, y>1 and this would be an absurdity.

The non validity of gr(f) ∩ b = ∅ consolidates the truthfulness of gr(f) ∩ b ≠ ∅ and therefore the transformation of "f probably has fixed points" from <u>conjectured-fact</u> to <u>fact</u>.

5.1.2 Analysis through the tools of the Abductive System

Excerpt	**Interpretation through the tools of the Abductive System**
f probably has fixed points The search of a justifying hypothesis needs the construction of a theory. In this case: to identify and explicate the properties of the fixed points. The need to broaden the cultural background in order to be able to build the hypothesis	CONJECTURE with role of answer to the problem, therefore it is a C-FACT. The C-FACT is created by a PHENOMENIC ACTION, guided by a didactical contract: "if the problem asks…" The statement describing the C-FACT is an UNSTABLE STATEMENT because Marco and Matteo don't believe the didactical contract sufficient to validate the statement.
The vertex of the squares on the paper sheet represents a fixed point  The graphic exploration continues	FACT created by a PHENOMENIC ACTION. It is expressed by a STABLE STATEMENT, in fact Matteo and Marco justify it through a visual impact that seems to be sufficient
The set of the fixed points is the bisector line  Now they have a new property in their cultural background	FACT created by a PHENOMENIC ACTION guided by the visual impact and by an unconscious consideration of the density of R^2. The fact is expressed by a STABLE STATEMENT, which is justified by: 1) the vertexes of the squares represent the fixed points; 2) cognitive jump: between two squares there are infinitely many others. The visual impact seems to be sufficient.

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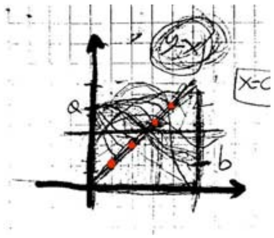
The continuous functions in [0,1] intersect the bisector line



Choice of a proving strategy: “proof by contradiction”. Probably guided by a didactical contract, because they recently saw such kind of procedure

</td>
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In this case we have two different stages.

First stage: THE ACT OF REASONING is created by a PHENOMENIC ACTION guided by a visual impact; and it is expressed by a UNSTABLE STATEMENT based on:

1) Continuous function in [0,1] onto [0,1] (given of the problem)
2) Bisector line as set of the fixed points (built by the student)
3) Continuous function in [0,1] means no gaps in the interval (student’s elaborated conception)

At this point, an ABDUCTIVE ACTION is accomplished: the C-FACT is reinterpreted as possible HYPOTHESIS corroborating the initial C-FACT (“the function has probably fixed point”; in fact if the function has a common point with the bisector line, then this point is fixed). The statement becomes an UNSTABLE ABDUCTIVE STATEMENT, unstable because Marco and Matteo do not believe the three aforementioned conditions sufficient to validate the hypothesis expressed by the statement.

Obs.: in the first phase it has been said that the <u>act of reasoning</u> represents a <u>c-fact</u>, relating its “instability” to that one considered when the act of reasoning takes the role of hypothesis. Nevertheless, we do not have to ignore the hypothesis that if the <u>act of reasoning</u> had been stopped at the first step, the visual impact could have been enough for Matteo and Marco, and then the act of reasoning would have been expressed by a <u>stable statement.</u>

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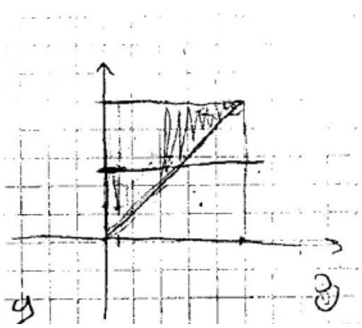
There exists a function continuous on [0,1] such that it doesn't intersect the bisector line. gr(f) $\cap$ b = $\emptyset$ Graphic exploration; adoption of transformational reasoning	PHENOMENIC ACTION guided by the structure of the proof by contradiction; this action creates a C-FACT expressed by an UNSTABLE STATEMENT.
gr(f) belongs to the upper triangle 	Creation of a HYPOTHESIS through an ABDUCTIVE ACTION guided by a visual impact. The hypothesis is stated by an UNSTABLE ABDUCTIVE STATEMENT in the sense that Matteo and Marco believe the visual impact to be insufficient to validate the hypothesis

In Marco and Matteo's protocol it is possible to find an abductive process both in conjecturing and evidencing process.

5.1.3 Daniele and Betta (limit problem)

<u>R1</u>: **D**: x_0+h...

<u>R2</u>: **B**: f (x_0)…

<u>R3</u>: **D**: *in my opinion it is the same thing… when you do the limit of the difference quotient, you do* $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ *…this minus this over h…*

He signs on the graph the vertical and the horizontal segments (see the red segments in Figure 10)

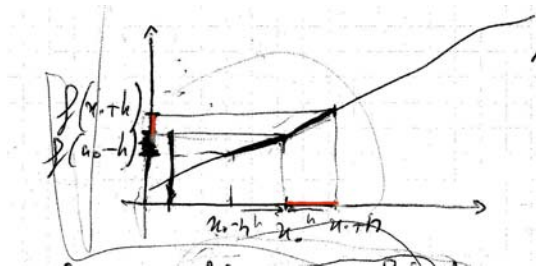


Figure 10: Daniele's graphic interpretation of $\dfrac{f(x_0+h)-f(x_0)}{h}$

R4: **D**: (note: he signs on the drawing done on the protocol, this | divided by this —)

R5: **B**: *because* $f(x_0 + h)$*...*

R6: **D**: *minus* $f(x_0)$*...is this*

R7: **B**: *Ah…OK…ours would be this (see the red segments in the figure 11) over 2h…it is the same thing…*

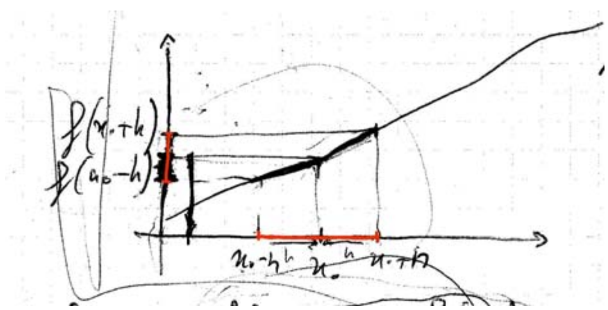


Figure 11: Graphic interpretation of $\dfrac{f(x_0+h)-f(x_0-h)}{2h}$

R8: **D**: *therefore…it would be* $h \to 0$*…how much is this?…eh…it will be the slope of the tangent line…*

R9: **B**: *namely…the first derivative*

R10: **D**: *in* x_0

Daniele draws a generic function f(x) and he signs on the axis x_0, x_0+h, x_0-h, f(x_0), f(x_0+h), f(x_0-h). The first tool he makes use of, is iconic; secondly he observes for a while the sign he produced, and then he says: "*…in my opinion it is the same thing";* namely, "doing $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$ is the same of $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ …"

The act of reasoning takes the role of answer to the problem, and it s a c-fact expressing a process. The Phenomenic action, which creates the c-fact, seems to be guided by a feeling, by a visual impact with the graphic representation that resembles the graphic situation met for the limit of the standard difference quotient.

The visual impact, though, is not sufficient to validate the act of reasoning, which represents a c-fact and it is expressed by an unstable statement:"*…in my opinion it is the same thing*".

The process follows with the search of a hypothesis validating the c-fact, to this extent:

a) There is a reinterpretation of the frame used for the standard difference quotient. Daniele translates the difference quotient as the ratio of the two segments <<this | divided by this — >> (see Figure 10)
b) Such interpretation is shifted to the present situation. Daniele states that the tools are the same: $\frac{f(x_0+h)-f(x_0-h)}{2h}$ is always the ratio between two segments (see Figure 11).

In this way the validating hypothesis is created: "the two limits use the same tools". Finally, the abductive action, which allowed the creation of the validating hypothesis, brings to a deductive process in the sense that being $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} =$ *$f'(x_0)$* and according to the validated fact that $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ are the same, then $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$ = $f'(x_0)$.

This hypothesis has probably also been generated by the kind of function sketched by Daniele. The choice of x_0 leads to a sort of symmetry related to $f(x_0)$; namely, $f(x_0+h)-f(x_0)$ and $f(x_0)-f(x_0-h)$ seem to be two segments of equal length.

At this point they explain their solution to me:

R16: **B:** *this is equal to this* (they indicate the two limits…)…*we did it graphically* (i.e., Betta indicates $\frac{f(x_0+h)-f(x_0-h)}{2h}$ and what they have highlighted graphically)

R17: **D**: *I mean, we do this…it would be the ratio between this difference* / *and this one* — *and in our case it would be the ratio between this difference* / *and this one* — *, therefore,* $x_0 + h - (x_0 - h)$ *that would be 2h…and this one that would be* $f(x_0 + h) - f(x_0 - h)$*...therefore, the limit for h that goes to zero would be…I mean both go to* x_0 (note: he shows it to me on the graph).

The validity of their hypothesis is justified graphically, and such a visual impact seems to be sufficient.

At this point I try to *provoke* Daniele and Betta and to insinuate in them the doubt about the adequacy of their graphical justification.

R21: **D**: *at an intuitive level, yes…but in my opinion it is not a rigorous justification*

R22: **I**: *why?*

R23: **D**: *because if somebody explained it to me in this way…I wouldn't…*

R24: **I**: *you wouldn't believe him?*

R25: **D**: *no…I mean…but it seems to me to know it only in this way…*

R26: **I**: (note: Daniele thinks)

R27: **D**: *eh yes…anyway it is correct…I mean, the difference quotient would be this chord …namely, it would be the tangent line of this angle, right? The difference quotient…therefore, for h that goes to zero, this…this chord…shrinks more and more till when it becomes a point and it is the tangent line in that point…in this case it is the same thing*

R28: **I**: *If you were told in this way…it would be enough for you? Would you be convinced if one of your classmates explained it to you in this way? Would you say….ah, OK…yes, yes…or would you have some doubts?*

R29: **D**: *we should write it down…*

R30: **I**: *how do you write such a thing?*

R31: **D**: *firstly, if I have an equation and I do the limits of the both parts…it is the same thing…*

Daniele employs again the graphic dynamics since when he tries an algebraic strategy. At this point they consider the equation

$$\frac{f(x_0+h)-f(x_0-h)}{2h} = \frac{f(x_0+h)-f(x_0)}{h}$$

This would justify the equality between the two limits. We have, then, the sub intended c-fact: $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$, and the justifying hypothesis $\frac{f(x_0+h)-f(x_0-h)}{2h} = \frac{f(x_0+h)-f(x_0)}{h}$ is built.

The consideration of this hypothesis is probably guided by a fact already acquired by Daniele and Betta, namely, if f(x) = g(x) $\forall x \in (x_0-\delta, x_0+\delta)$ then $\lim_{x\to x0}$ f(x) = $\lim_{x\to x0}$ g(x). It follows a series of algebraic manipulations based on an erroneous starting idea, namely, Daniele and Betta in proving the equality between the two expressions start exactly from $\frac{f(x_0+h)-f(x_0-h)}{2h} = \frac{f(x_0+h)-f(x_0)}{h}$

<u>R32</u>: **B**: *therefore, if you prove that this is equal to this* (namely, $\frac{f(x_0+h)-f(x_0-h)}{2h}$ and $\frac{f(x_0+h)-f(x_0)}{h}$)

<u>R33</u>: **D**: eh…therefore…yes but…I must… it would be

$$2\,\frac{f(x_0+h)-f(x_0)}{h} = \frac{f(x_0+h)-f(x_0-h)}{2h}\,2$$

And they simplify in the following way

$$2\,\frac{f(x_0+h)-f(x_0)}{\cancel{h}} = \frac{f(x_0+h)-f(x_0-h)}{\cancel{2h}}\,\cancel{2}$$

<u>R34</u>: **I**: *but then you have already given for sure that this and this one are equal…*

<u>R35</u>: **D**: *ehm…yes…*

<u>R36</u>: **I:** *no, you have to prove it. I thought you would want to prove that*

$$\frac{f(x_0+h)-f(x_0-h)}{2h} = \frac{f(x_0+h)-f(x_0)}{h}$$

<u>R39</u>: **D**: *yes…but you are right! I already thought to be true the equality…then, I looking for…no, no…*

After some further algebraic manipulations, Daniele goes back to the graph and he realizes that the line connecting the points A(x_0+h, f(x_0+h)) and B(x_0-h, f(x_0-h)) and the one connecting A(x_0+h, f(x_0+h)) and C(x_0, f(x_0)) have one point in common; therefore, proving the equality between the two difference quotients would mean to prove the parallelism between two lines that go through one same point, that is impossible. The graphic tool becomes an important means to invalidate the previously built hypothesis, namely, the equality: $\frac{f(x_0+h)-f(x_0-h)}{2h} = \frac{f(x_0+h)-f(x_0)}{h}$

Daniele and Betta at this point realize that the error has been conveyed by the drawing of a particular function, such that $|f(x_0+h)-f(x_0)| = |f(x_0)-f(x_0-h)|$

<u>R69</u>: **D**: *we did a drawing that misled us*

<u>R77</u>: **D**: *but now neither the graphic one convinces me anymore…because we used the symmetry respect to f(x_0)…no, no…that one is true*

<u>R79</u>: **I**: *what has been the conjecture brought up by the graph? Therefore…from the graph you said…probably is f'(x_0)*

Daniele starts doubting about the graphic justification too, but then he realizes that the important thing is the passage to the limit that brings to the same tangent line in (x_0, f(x_0)) They remain convinced that such a limit represents the first derivative of f in x_0, that is the rate of change of the tangent line; they remain, therefore, considering the fact that the two limits are the same; they only abandon the previously built hypothesis, because it revealed to be false. It is interesting to observe that the non-validity of the hypothesis has undermined the conviction of the fact

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

only for a while, after which the graphic approach has prevailed.

The following step is to retake the algebraic manipulation, this time they add and subtract $f(x_0)$, probably with the aim to obtain partially the expression of the standard difference quotient; they separate the two expressions getting

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{2h}+\lim_{h\to 0}\frac{f(x_0)-f(x_0-h)}{2h}$$

from which they write

$$\frac{f'(x_0)}{2}+\frac{f'(x_0)}{2}.$$

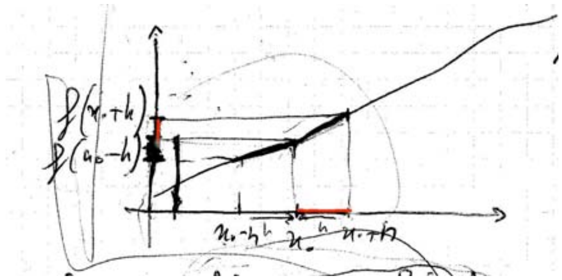
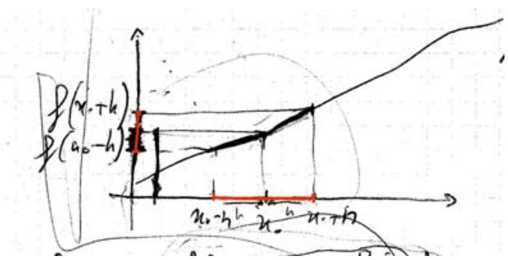
At the moment I asked them to explain me why the

$\lim_{h\to 0}\frac{f(x_0)-f(x_0-h)}{h}$ was $f'(x_0)$

they answered me they had seen it graphically.

5.1.4 Analysis through the tools of the Abductive System

Excerpt	**Interpretation through the tools of the Abductive System**
In my opinion it is the same thing… Namely, doing $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$ is the same as $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ … Search of a validating hypothesis	CONJECTURE with the role of answer to the problem; therefore, C-FACT. The C-FACT expresses a process, and it is created by a PHENOMENIC ACTION guided by a feeling, by a visual impact with the graphic representation met for the limit of the standard different quotient. The statement describing the C-FACT is an UNSTABLE STATEMENT because the visual impact seems to be insufficient to validate the act of reasoning.

 The two limits use the same tools 	Creation of a HYPOTHESIS through an ABDUCTIVE ACTION guided by the reinterpretation of the frame used for the standard difference quotient: Daniele translates the difference quotient as the ratio between the vertical and horizontal segments (see the two figures) and he shifts such interpretation to the present situation. The act of reasoning seems to be expressed by a STABLE STATEMENT since the graphical justification results sufficient for them. Probably such a kind of hypothesis has been also generated by the kind of function sketched by Daniele. The choice of x_0 leads to a sort of symmetry related to $f(x_0)$; namely, $f(x_0+h) - f(x_0)$ and $f(x_0) - f(x_0-h)$ which seem to be two segments of equal length.
A new phase starts. I provoke Daniele and Betta with the aim to generate the doubt about the adequacy of their graphical justification	

$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} =$ $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ Search of a validating hypothesis	The C-FACT is not changed; and the PHENOMENIC ACTION is always guided by a visual impact. The act of reasoning is expressed by an UNSTABLE STATEMENT.
$\frac{f(x_0+h)-f(x_0-h)}{2h} = \frac{f(x_0+h)-f(x_0)}{h}$ They start with algebraic manipulation to prove the equality. After several attempts, they go back to a graphic exploration and they find out that such equality would confirm the parallelism of the two lines; this is impossible since both go through the point (x_0+h, f(x_0+h)). This brings the two students to refute the aforementioned hypothesis. Nevertheless, they go back to the graphic exploration and their c-fact does not change, because the graphic dynamics reinforce their conviction that when x goes to x_0 the line becomes the tangent line, therefore the limit represents the first derivative like the limit of the standard difference quotient. What changes is the approach to prove the c-fact, with a new manipulation of the starting expression.	Creation of a HYPOTHESIS through an ABDUCTIVE ACTION probably guided by a fact already acquired, namely if $f(x) = g(x) \quad \forall x \in (x_0-\delta,\ x_0+\delta)$ then $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x)$. the hypothesis is expressed by an UNSTABLE STATEMENT.

The algebraic manipulation brings to the expression $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{2h}+\lim_{h\to 0}\frac{f(x_0)-f(x_0-h)}{2h}$ with the construction of a new conjecture.	
$\lim_{h\to 0}\frac{f(x_0)-f(x_0-h)}{h}$ = f'(x_0)	This act of reasoning take the connotation of FACT in the sense that they justify it through the graphical interpretation as they did previously with the initial expression and the graphic interpretation this time is enough. A STABLE STATEMENT therefore expresses the fact.

5.1.5 Francesca and Daniele (fixed point problem)

Francesca and Daniele visualize immediately on a graph the set of the fixed points as the bisector line of the first and third orthant; the property is already present in their cultural background: “the bisector line is the set of the fixed points”. The drawing itself probably suggests the characterizing property of the fixed points which is, according to Francesca, *the belonging to the bisector line.*

Therefore, there is the consideration of a fact: *the set of the fixed points is represented by the bisector line of the first and third orthant.* The fact is implicitly expressed by a stable statement, in the sense that Francesca and Daniele consider this fact already acquired; as it would be already part of their cultural background. The subsequent step is the consideration of a further fact that expresses a property: *to be a fixed point means to belong to the bisector line;* in this case too, it seems not to need further proof, being the consequence of the definition of the fixed points.

<u>R1</u>*:* **Fr.:** there must be an intersection between the function and the bisector line

R3*:* **Fr:** if there is the fixed point there absolutely must be the intersection with the bisector line.

From R1 and R3 it seems a positive answer has already been taken into consideration, namely, *the function has a fixed point.* Such an attitude may be conveyed by a didactical contract: "if they ask, probably there will be one"; or it is simply a choice at the 50 per cent. Even in this case, like in Matteo and Marco's protocol, the first step is represented by the attempt to give an immediate answer from which to proceed. Considering thefact of the presence of a fixed point, the subsequent step is represented by the construction of a hypothesis, which may validate the fact, and which takes the aspect of conjecture and therefore it represents a c-fact.

Francesca states: "there exists an intersection with the bisector line"; we are in front of an abductive statement built through a deductive process, namely, it seems to be led by the characterization (according to Francesca's cultural background) of the fixed point, that is translated by Francesca from $f(x) = x$ to $y = x$. That means, if the function has a fixed point, such a point, for its characterization, must stay on the bisector line (note: my interpretation was confirmed later by Francesca). Therefore, the abductive action may have been guided by the following deductive process:

$\forall P \in b \to P$ is fixed   (b is the bisector line)

$Q \in b$

Q is fixed   *modus ponens*

Summarizing:

1st step: consideration of the set of the fixed points as the bisector line

2nd step: construction of the theory: "being a fixed point means to belong to the bisector line" (happened by deduction)

3rd step: consideration of the c-fact "$\exists$ a fixed point"; guided by a didactical contract or by a simple choice at 50%. It is expressed by an unstable statement (such reasons seem to be not sufficient to legitimate the conjecture); in fact Francesca looks for a hypothesis that could validate the answer.

4th step: consideration of a hypothesis: “the function intersects the bisector line”, validating the c-fact. This second statement would explain the existence of a fixed point. In fact, the belonging of a point to a bisector line is the equivalent, for Francesca, of being a fixed point; therefore, if the function intersects the bisector line, the function has a point in common with this line and therefore the first function has a fixed point too.

The abductive statement is an unstable statement, because thus far, Francesca does not know if the function intersects the bisector line.

R5: **Fr***: if there weren’t (fixed points) it (the function) would stay all over or all under the bisector line…the only case would be if the bisector line were the asymptote of the function…*

R6: **Dan**: *but it is not possible*

R7: **Fr**: *…but it is not possible because it is continuous…*

R8: **Dan***: it is not possible because 1 is between…I mean…the function in 1 exists…that is, here it is included…(**ndr:** he writes a square parenthesis on 0 and on 1 on the x-axis and he does the same thing on the y-axis)*

R9: **Fr**: *therefore the bisector line cannot be an asymptote, and then if it is not an asymptote it must cross it for sure…*

R10: **Fr** (talking to I) *probably we answered…if A is a fixed point it must have an intersection with the bisector line…the only case for the contrary is if the bisector line were the asymptote of the function…but, if the function is defined from [0,1] to [0,1] included…the function is defined in 1 too, therefore at the most the point is (1,1) or it crosses it.*

The proving process begins with the denial of the fact: “if there weren’t fixed points…”, and it follows with the proof by deduction that it is not possible because of the continuity of f in [0,1] onto [0,1].

This proof doesn’t seem to be sufficient; perhaps they feel the possible fallacy of the visual impact they used in the proving process; and therefore there is the search of something “mathematically acceptable”. The sign, at this point, plays an important role due to the fact that the bisector line and the sketch of some possible functions evoke the *Theorem of the Zeros*.

R16: **Dan**: (he draws several functions, then he realizes that it is not like that) *anyway, there is one fixed point for sure…if it must take all the values and if we make it start from here…if it must take all the values it must start from this point…from this…this…because it can't come back…to take all the values it must start from the maximum up to the minimum…if we think of that theorem where if you have a point here and one here it must go through here, for sure* (see Figure 12).

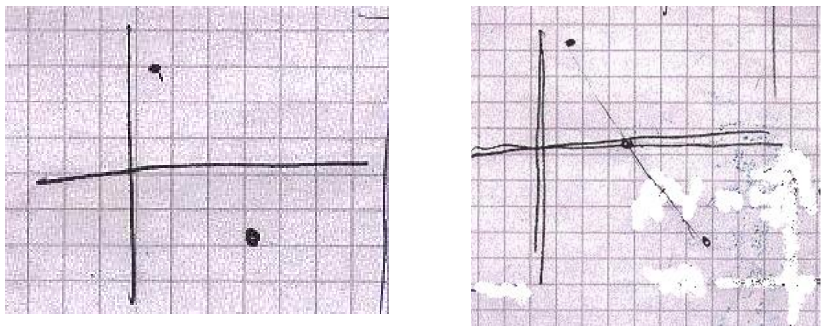


Figure 12: Daniele's drawings

R17: **Fr**: *it is the Theorem of the Zeros…*

[…]

R27: **Fr**: *oh yes…instead of the x-axis we have a line*

R28: **Dan**: *the bisector line…*

R42: **Dan**: (talking to me) *is it enough in this way?…I mean, if it is a proof that can be accepted or not (Daniele explains the proof)…by the moment that it must take all the values of the Image, $a > x$ $b < x$…(he corrects himself) $f(a) > x$ $f(b) < x$* (see Figure 13).

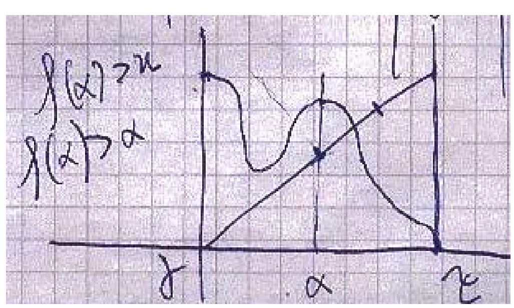


Figure 13: Graphic aid for the proof

R43**: I**: *what is x?*

R44: **Dan**: *x is the bisector line, otherwise if b were here it could not take all the values of the Image because the function could not do like this (and he traces a vertical line)…*

R50: **Fr**: *ah no…anyway, the theorem of the zeros shifted up…for example this is the line x = 2 there is necessarily a point f(x) = 2 and therefore the same thing if we take the bisector line as the line…there is a point that is over…one that is under…there must exist necessarily a point that lies on the bisector line*

R51: **I**: *Why?*

R52: **Fr**: *because the function is continuous*

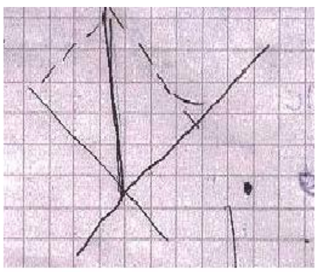
R54: **Dan**: *If I divide the bisector line in several intervals…*

The proof continues with the attempt to adapt the proof of the theorem of the zeros to the new situation (see the attached transcripts at the end of the work); the proof will remain "technically incomplete", but the conviction that the present situation is a modified situation of the *Theorem of the Zeros*, gives them the certainty of the existence of the fixed point. In this phase we can observe the ability of adapting a proving process that brings to light an internalization (Harel) work of a cognitive process.

It is important to observe that in the evidencing phase we can find a sort of *abductive attitude*, in the sense that Francesca and Daniele justify their idea to use the structure of the proof of the *Theorem of the Zeros* because the graphic situation is analogous and the x-axis is replaced by the bisector line.

Therefore, there is the construction of a C-FACT: *it is possible to use the structure of the proof of the theorem of Zeros*, and the construction of the justifying HYPOTHESIS: *analogous graphic situation, and the x-axis is replaced by the bisector line.*

5.1.6 Analysis through the tools of the Abductive System

Excerpt	Interpretation through the tools of the Abductive System
The set of the fixed points is represented by the bisector line of the I and III orthant  Need to translate such a fact in a sort of theory or regularity	FACT created by a PHENOMENIC ACTION guided by the necessity of visualization. The statement describing the fact is a STABLE STATEMENT justified by an already acquired knowledge, or just an immediate translation of f(c) = c into y=x, and its graphic representation seems to be enough to justify the ACT OF REASONING.
Being a fixed point is equivalent to belong to the bisector line of the I and III orthant	CREATION OF A "THEORY" OR A REGULARITY created by a PHENOMENIC ACTION guided the interiorization (Harel) of a cognitive process: <u>deduction</u>. The statement describing the FACT is a STABLE STATEMENT, because the consideration of the bisector line as the set of the fixed points and the deduction $\forall$ P fixed $\rightarrow$ P$\in$ b Q fixed __________ Q $\in$ b (MODUS PONENS) seems to be sufficient.

The function has a fixed point ↓ Now they need to find a hypothesis that could justify the c-fact	A PHENOMENIC ACTION, guided by a didactical contract (if the problem asks such a question, probably the function has a fixed point; or just a choice at 50%) generates a CONJECTURE with the role of the answer to the problem, therefore it becomes a C-FACT. The C-FACT is expressed by an UNSTABLE STATEMENT, because Francesca and Daniele don't consider the didactical contract or a choice at 50% to be good enough to justify the conjecture.
Exists an intersection between the function and the bisector line ↓ The need of validating the hypothesis takes place	HYPOTHESIS created by an ABDUCTIVE ACTION guided by the interiorization of a cognitive process: <u>deduction</u> $\forall$ P b(P) $\rightarrow$ P fixed $\exists$ Q $\in$ gr(f) such that b(Q) $\exists$ Q $\in$ gr(f) such that Q is fixed (MODUS PONENS) The HYPOTHESIS is expressed by an UNSTABLE ABDUCTIVE STATEMENT, because she knows that if it were verified, then it would legitimize the existence of the fixed point. But she doesn't know if the function satisfies such a condition for sure.
It is possible to use the same structure of the proof of the Theorem of Zeros ↓ They feel the need to justify it	C-FACT created by a PHENOMENIC ACTION guided by a visual impact and by their knowledge of the theorem and of its proof.

There is an analogous graphic situation, and the bisector line replaces the x-axis.	ABDUCTIVE ACTION, which builds the HYPOTHESIS justifying the C-FACT. It is expressed by a STABLE STATEMENT because the graphic evidence concerning the analogies with the known theorem seems to be sufficient.

5.1.7 Alice and Roberta (fixed point problem)

R2: **R:** *fixed point on the bisector line and therefore…*

Roberta and Alice immediately state that the fixed point lies on the bisector line and then they draw the line. A property like "P fixed point ⇒ P ∈ bisector line" is already present in their cultural background.

R3: **A:** (she draws the bisector line) *therefore this is the (1,1) and (0,0).* The idea becomes sign (Figure 14)

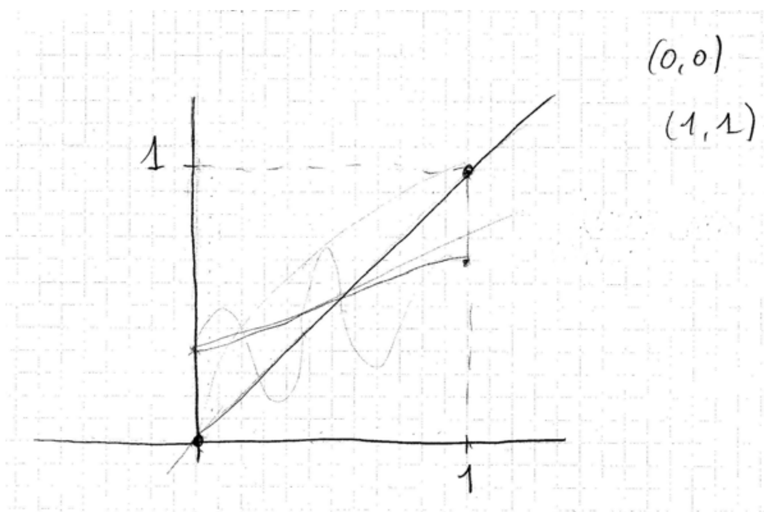


Figure 14: Alice's drawing of the bisector line

R4: **R:** *The fixed point must be between these two points* …(and she signs the two points (0,0) and (1,1) going along the bisector line)

Consideration of a fact: "the fixed point is on the segment with end points (0,0) and (1,1)." Such fact seems to be legitimated by the property previously considered (namely, the fixed points lay on the bisector line), and by the domain of f which is [0,1].

<u>R5</u>**:** **A**: *Exactly…but it could have only these two points [(0,0), (1,1)]; if it were in this way* (and she signs a concave function over the bisector line) *therefore there is a fixed point for sure, because there are these two points of the bisector line* (and she signs (0,0) and (1,1))

Consideration of two new facts:

1) "The function has for sure two fixed points which are (0,0) and (1,1)";
2) "And it has only those if the graph of f is done in a particular way".

Both facts seem to be expressed by means of stable statements; in the first case Alice has taken into consideration, a-priori, that the function starts from (0,0) and finishes into (1,1); probably she has been conditioned by a visual impact with the graph she has produced, and by the definition of the function in the problem. The second act of reasoning is expressed by a stable statement, which is supported by a graphic impact.

<u>R6</u>**: R:** *eh…no, because the function could start from here and from here.*

What is a stable statement for Alice is an unstable statement for Roberta, who immediately refutes Alice's assertion showing that the function does not necessarily starts from the point (0,0), but it can start from any point of the segment whose extremes are (0,0) and (1,1).

<u>R7</u>**: A:** *you are right, it is true; it is defined from 0 to 1…*

<u>R8</u>**: R:** *oh yes...the function starts from 0 and then there is a point here for sure (she underlines the segment from 0 to 1 on x-axis) and it arrives at x=1, therefore there is also a point here for sure (she underlines the side of the square of vertexes (1,0) and (1,1))* (see figure 4).

<u>R9</u>**: A:** *oh right...then it has to intersect the bisector line for sure…suppose that it does like that…*

The newly produced sign leads to the construction of a new fact: "The function intersects for sure the bisector line." This fact is expressed by a stable statement justified by knowledge already possessed by the students, namely, "the continuous functions don't have gaps in their graph."

<u>R10</u>: **R:** *hmmm…the function must have a fixed point for sure…because it has to pass from here to there.*

Construction of a fact "The function had a fixed point for sure." The fact is justified because the function intersects the bisector line.

Both Alice and Roberta seem to be arrived at the conclusion that the function definitely has a fixed point; namely their act of reasoning is expressed by a stable statement.

From here the process becomes an evidencing process, in the sense that they don't try to establish the stability of their statement, since it is already stable for them, but they try rearrange their "proof" in a way that can be considered acceptable by the others.

R27**: A/R:** *then the function must start from 0 and have f(x) on this side and arrive at the point of abscissa x=1 and f(x) on this side then…there is the bisector line that goes through (0,0) and (1,1)*

R33**: A:** (Alice writes) *then* $P(0, 0 \leq y \leq 1)$ *because the domain…*

R34**: R:** *it is defined from 0 to 1*

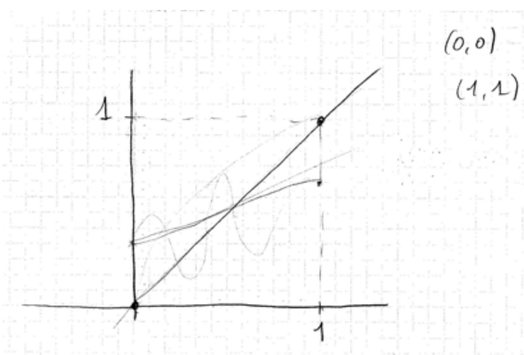
R37**: A** (Alice writes) *it must exist too…*$P_1 (1, 0 \leq y \leq 1)$*, I would start with the limit cases, P(0,0) and* P_1 *(1,1) or when* (she goes with her finger from the point (0,0) along the segment 0-1 on the y-axes, and she does the same with the point (1,1) downwards)

At this point they write on their protocol:

If the function f(x) goes through P(0,0), a fixed point is P; there could exist other fixed points in the case that the function intersects the bisector line.

In the same way, if the function goes through the point P(1,1). In all other cases the function will have to go through a point with abscissa 0 and a point of abscissa 1 (for hypothesis). In these cases the ordinate of the point with abscissa 0 will have to be $0 \leq y \leq 1$*, and the ordinate of the point with abscissa 1 will have to be* $0 \leq y \leq 1$*. Being the function continuous for any path satisfying the aforementioned conditions will have to intersect the bisector line in at least one point (on the bisector line lie all the fixed points).*

5.1.8 Analysis through the tools of the Abductive System

Excerpt	Interpretation through the tools of the Abductive System
The fixed point must be between these two points 	FACT created by a PHENOMENIC ACTION guided by the necessity of visualization. The statement describing the fact is a STABLE STATEMENT justified by an already acquired knowledge, namely, “the fixed points are on the bisector line”, as they claim in R1.
The function has for sure two fixed points which are (0,0) and (1,1)	This act of reasoning owns a different role for Alice and Roberta. According to Alice, this is a FACT, expressed by a STABLE STATEMENT justified by her conviction that the functions, as defined in the problem, starts from (0,0) and ends in (1,1). According to Roberta the statement expressed by Alice is an UNSTABLE STATEMENT, since the function does not necessarily starts from (0,0) and ends in (1,1); and for this reason it is a C-FACT, which Roberta immediately refutes. Roberta could contest the statement, probably because she was able to use a transformational reasoning (in sense of Harel’s) which allowed her to imagine different situations of the graph of the function f.

The function intersects the bisector line for sure.	FACT, created by a PHENOMENIC ACTION, guided by a graphic exploration. The FACT is expressed by a STABLE STATEMENT justified by a knowledge already possessed by the students, namely, the “continuous functions don’t have gaps in their graphs.”
The function must have a fixed point for sure	FACT created by a PHENOMENIC ACTION guided by a deductive process: Any point on the bisector line is fixed The function has a point on the bisector line (seen before) ——————————————— The function has a fixed point (MODUS PONENS) The FACT is expressed by a STABLE STATEMENT, since the preceding visual impact (regarding the intersection between the function and the bisector line) and the deductive process mentioned above seem to be sufficient to justify the act of reasoning.

5.1.9 Francesca and Serena (limit problem)

R1: **S**: *h goes to zero…x_0+h…*

They immediately draw a graph tracing on the axis x_0, $x_0 + h$, $f(x_0)$, $f(x_0+h)$…

R2: **S**: *$f(x_0+h)$*

She looks at it on the graph

R3: **S**: *when h $\rightarrow$ 0 this gets closer here and also $f(x_+h)$*

[…]

<u>R7</u>: **S**: *anyway, this difference goes to zero…and if we separate them?…* $\frac{f(x_0+h)}{2h}$ …

$f(x_0+h) \to f(x_0)$…

There is the creation of a fact, but it seems this fact does not convince them…probably they abandon such a idea and think instead of a strategy: "to separate the two addends". The choice is probably suggested by a previous knowledge related to the expression of the standard difference quotient; in fact Francesca adds a and subtracts $f(x_0)$ with the aim of obtaining a part of the expression of the standard difference quotient.

<u>R8</u>: **F**: $\frac{f(x_0+h)}{h} - \frac{f(x_0)}{h}$ *and then we add it…*

<u>R9</u>:**F&S**: *let us write it down better:* $\lim_{h\to 0} \frac{1}{2}\left(\frac{f(x_0+h)}{h} - \frac{f(x_0)}{h}\right) + \frac{1}{2}\left(\frac{f(x_0)}{h} - \frac{f(x_0-h)}{h}\right)$

<u>R10</u>: **F**: *this* (referring to the first parenthesis) *is our $f'(x_0)$ therefore* $\frac{1}{2}f'(x_0)$

<u>R11</u>: **S**: *that thing there* (referring to the second parenthesis)…

<u>R12</u>: **F**: *it will be a difference quotient as well…because if you look at the drawing…from this you take off this and divide by h; from that you take off this and you subtract h, therefore the difference should be the same thing…*

At this point there is the creation of a new fact, probably guided by a visual impact very similar to the standard difference quotient. There is also the construction of a justifying hypothesis, which is based on a graphic interpretation of the difference quotient.

<u>R13</u>: **S**: *then…1/2 $f'(x_0)$ – 1/2* $\lim_{h\to 0}\left(\frac{f(x_0)}{h} - \frac{f(x_0-h)}{h}\right)$ *this goes to zero…*

<u>R14</u>: **F**: *hmmmm….*

<u>R15</u>: **S**: *in my opinion is wrong…ah…but wait…here there is –h therefore this becomes +…then $f'(x_0)$…*

<u>R19</u>: **F**: *yes…also because basing on my intuit I would have said that the limit would go to $f'(x_0)$….therefore* $\left(\frac{f(x_0)}{h} - \frac{f(x_0-h)}{h}\right)$ *is the difference quotient*

Francesca explains to me what they did

R20: **F**: We did it very algebraically…*and we said…first we add* $\left(\frac{f(x_0)}{h}\right)$ *and then we subtract it…first we take out 1/2 …* $\frac{1}{2}\left(\frac{f(x_0+h)}{h}-\frac{f(x_0-h)}{h}\right)$ *I add and subtract* $\frac{f(x_0)}{h}$ *therefore here taking it out, I have exactly the difference quotient, thus I have f' (x_0) here…*

R22: **I:** *here can I say that it is f'(x_0)?*

R23: **F:** $\left(\frac{f(x_0-h)-f(x_0)}{h}\right)$ *let us change the signs…* $-\frac{1}{2}\left(\frac{f(x_0)-f(x_0-h)}{-h}\right)$ and we said…

R24: **S:** *that the difference quotient can be* $\left(\frac{f(x_0+h)-f(x_0)}{h}\right)$ *but also* $\left(\frac{f(x_0-h)-f(x_0)}{-h}\right)$

R25: **I:** *Why?*

R26: **S:** *because h goes to zero therefore –h goes to zero and thus even this is f'(x_0), then* $\frac{1}{2}f'(x_0)+\frac{1}{2}f'(x_0)=f'(x_0)$

5.1.10 Analysis through the tools of the Abductive System

Excerpt	Interpretation through the tools of the Abductive System

…anyway, this difference goes to zero(she refers to the limit of the problem) The conjecture does not convince Serena, even though she has formulated it. There is the necessity to try another way. This brings Serena and Francesca to add and subtract $f(x_0)$ and it leads to the expression $\lim_{h\to 0}\frac{1}{2}\left(\frac{f(x_0+h)}{h}-\frac{f(x_0)}{h}\right)+\frac{1}{2}\left(\frac{f(x_0)}{h}-\frac{f(x_0-h)}{h}\right)$	C-FACT created by a PHENOMENIC ACTION guided by a graphic exploration that shows the getting closer to the same point, for $h\to 0$, of $f(x_0-h)$ and of $f(x_0+h)$. The statement describing the act of reasoning is an UNSTABLE STATEMENT since for some feeling this conjecture doesn't convince Serena.
...that thing there (referring to the second parenthesis)… *it will be a difference quotient as well* Search of a validating hypothesis	C-FACT created by a PHENOMENIC ACTION guided by a visual similarity with the standard difference quotient expression. The ACT OF REASONING is expressed by an UNSTABLE STATEMENT, since the visual analogy seems to be not enough.
…if you look at the drawing…from this you take off this and divide by h; from that you take off this and you subtract h, therefore the difference should be the same thing… This hypothesis can be translated as follows: *Both expressions represent the same procedure*	HYPOTHESIS created by an ABDUCTIVE ACTION guided by the graphic interpretation of the difference quotient. The HYPOTHESIS is expressed by a STABLE ABDUCTIVE STATEMENT since the graphic interpretation seems to be enough to justify the hypothesis.

5.1.11 Alice and Marco (limit problem)

R2: **A:** *at the end…it is the difference quotient…only that there is 2h instead of h…*

There is the creation of a fact conveyed by a visual analogy with the standard difference quotient.

R4: **A:** *no…wait…*

R7: **M**: *it is similar to the difference quotient…then…the difference quotient is…*(they think for a while and then they conclude) $\frac{f(x_0+h)-f(x_0)}{h}$*…yes…yes it is similar to…but there is not* $f(x_0-h)$

The fact is modified, and it becomes "it is similar to the difference quotient"

At this point they write $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$

R9: **A:** *I write also the difference quotient.*

R12: **M:** o*h yes...in other words we have the limit of two functions, I mean, the limit of* $\frac{f(x_0+h)}{2h}$ *minus the limit of* $\frac{f(x_0-h)}{2h}$ *and we cannot say that is the limit of the difference, so to speak, we take the result of this…*

R13: **A:** *but with the limit...what we arrive to say? Because…at the end…we know how to calculate this limit…we know that the function is defined and differentiable, therefore we know that is continuous, then we don't need to do all the calculation of the limit…*

In these first lines, they start saying that the expression is similar to the difference quotient because of a visual analogy. There is, then, a recall to their cultural background. They manipulate the expression and examine what they can say about each limit. At the end they conclude such an approach will not bring them to anything concrete.

R17: **A**: *You know what we can do? In* $\frac{f(x_0+h)-f(x_0)}{h}$ *there was the graph to show that it was the slope…*

There is the search in their own cultural background of what they learnt about the $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$

R18: **M***: yes…of the line…*

R19: **A**: *perhaps this is related to the slope but shifted up or down…*

They draw a function and reproduce on the graph the difference quotient (see Figure 15) and they build a new fact guided by an already acquired knowledge.

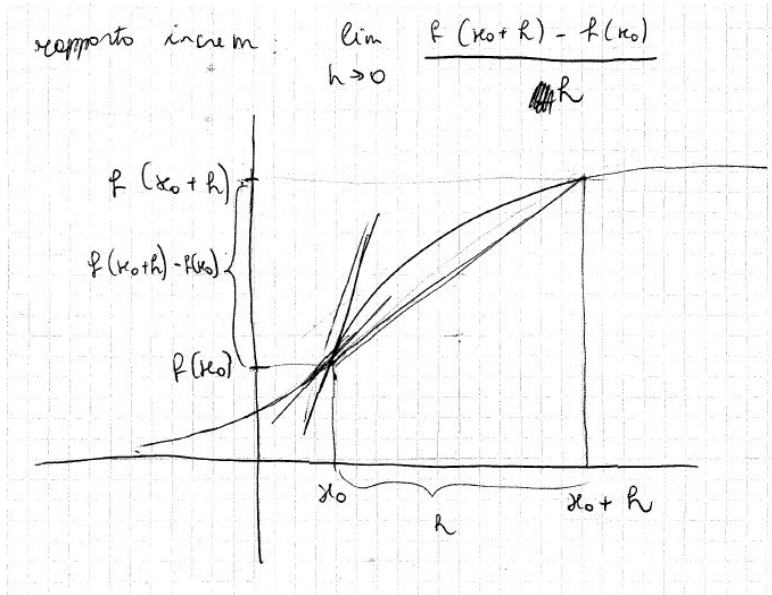


Figure 15: Graphic interpretation of the limit of the standard difference quotient

R20: **A**: *I mean…when h $\to$ 0…do you remember the graph?*

[…]

R25: **A**: *then…when h $\to$ 0…oh yes…this becomes the tangent line in this point here…*(Figure 16)

[…]

R31: **A**: *now let us try to draw this (* $\frac{f(x_0+h)-f(x_0-h)}{2h}$ *)* (see Figure 16)

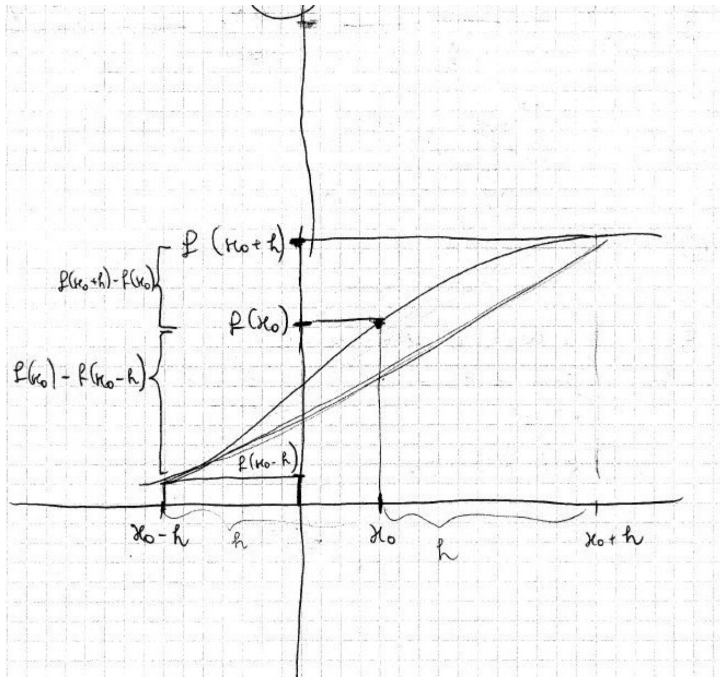


Figure 16: Representation of $\dfrac{f(x_0+h)-f(x_0-h)}{2h}$

R33: **A**: *in my opinion this could work as a difference quotient…*

Based on the graphic exploration, the construction of a new fact occurred: “$\dfrac{f(x_0+h)-f(x_0-h)}{2h}$ works as difference quotient as well”.

R34: **M**: *but the difference quotient is the slope of the tangent line…*

R35: **A**: *yes…*

R36: **M**: *and there, it goes…here what does this (*$\dfrac{f(x_0+h)-f(x_0-h)}{2h}$*) represent?*

R41: **A**: *It could represent the slope of the tangent line…*

Again the construction of a new fact, namely “the $\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0-h)}{2h}$ could be the slope of the tangent line”.

R42: **M**: *the tangent line in which point…?*

<u>R43</u>: **A**: *We need to see in which point…then, if h goes to zero…let us see what happens when h goes to zero…it means that…here there is a distance of 2h…between* $x_0 + h$ *and* $x_0 - h$

<u>R45</u>: **A**: *when h goes to zero, this becomes zero and goes to* x_0*, this one becomes zero and goes to* x_0*…therefore all the values go to* x_0*…while here (she refers to* $\frac{f(x_0+h)-f(x_0)}{h}$*)…too…at the end they always go to* x_0*…because the numerator when h goes to zero goes to…wait…goes to zero…*

<u>R46</u>: **M:** *here* (referring to the expression $\frac{f(x_0+h)-f(x_0)}{h}$) *it goes to…zero…oh…OK*

<u>R47</u>: Alice signs on the y-axis $f(x_0+h)-f(x_0)$ and $f(x_0)-f(x_0-h)$ (see figure 5)

<u>R48</u>: **A**: *then…here we have* $\frac{f(x_0+h)-f(x_0-h)}{2h}$ *...* $f(x_0-h)$ *is equal to* $f(x_0+h)$*-…*

<u>R49</u>: **M**: *minus* $\frac{f(x_0)}{h}$*...*

At this point they try to find graphically $f(x_0$-h)…but they realize they don't arrive at anything…

It is important to underline that the graphic exploration has led Alice and Marco to state that the limit would represent the slope of the tangent line in x_0; such a statement seems to be expressed by an unstable statement, since Alice feels the necessity to justify it algebraically. The algebraic exploration, though, suggests a different result and this is sufficient to make them to forget their graphical conclusion. Therefore, an algebraic manipulation takes place in order to obtain some kind of expressions similar to the standard difference quotient.

<u>R51</u>: **A**: *but we can write it as...I mean the limit of this one...* $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$ *...as a matter of fact we know the numerator, we can write it as addition and subtraction of limits in such a way to have inside of the expression* $\frac{f(x_0+h)-f(x_0)}{h}$

<u>R52</u>: **M**: *OK...you take out* $\frac{1}{2}$*…*

Alice writes $\frac{1}{2}\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{h}$

R54: **M**: *do you want to have the difference quotient?*

At this point they think for a while…they observe the graph they made…(see Figure 16)

R56: **A**: *we could write…(she adds and subtracts f(x_0)) and then we separate it…* $\frac{1}{2}(\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}-\lim_{h\to 0}\frac{f(x_0-h)-f(x_0)}{h})$ *the first become f'(x_0) and the second one?…I don't know…*

R57: **M**: *isn't it the difference quotient with the difference that there is a minus? Therefore it is the same thing but considered at the other side…*

Creation of a new fact: "the expression $\frac{f(x_0)-f(x_0-h)}{h}$ is like the standard difference quotient with the only difference being that there is a minus before.

R64: **A**: *then it could be zero…I mean…in both cases you arrive at the slope of the tangent line here. Therefore, it is the same thing of doing the slope of the tangent line here, minus the slope of the tangent line always here…*

R68: **M**: *yes. Zero.*

The graphic interpretation has completely disappeared; they see in both limits the slope of the tangent line in x_0 but they don't relate the algebraic interpretation with the graphic one; the algebraic impact prevails, and they don't realize that a limit equals to zero would imply a tangent line parallel to the x-axis which would be in contrast with their graphic representation.

5.1.12 Analysis through the tools of the Abductive System

Excerpt	**Interpretation through the tools of the Abductive System**
At the end …it is the difference quotient…only that there is 2h instead of h…(referred to $\frac{f(x_0+h)-f(x_0-h)}{2h}$) Very soon Alice realizes that the expression is not the difference quotient, but just similar to it. The previous fact is transformed into	Initially the act of reasoning seems to be represented by a FACT created by a PHENOMENIC ACTION guided by a visual analogy with the standard difference quotient. The fact is expressed by a STABLE STATEMENT where the analogy with the standard difference quotient seems to be enough.
It is similar to the difference quotient.	FACT created by a PHENOMENIC ACTION conveyed by a visual analogy with the standard difference quotient; such analogy seems to be enough to justify the STABLE STATEMENT expressing the fact.
Perhaps this is related to the slope…but shifted up or down…(referred to $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$)	C-FACT created by a PHENOMENIC ACTION conveyed by a recall to an already acquired knowledge about the relationship between the $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ and the slope of the tangent line. They shift this relationship to the $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$ imagining a sort of translation of the line. The act of reasoning is expressed by an UNSTABLE STATEMENT since the internalization (Harel) of the graphic interpretation of the limit of the standard difference quotient seems not sufficient to justify the C-FACT

In my opinion this could work as a difference quotient There is an evolution in the act of reasoning. From stating that $\frac{f(x_0+h)-f(x_0-h)}{2h}$ *is the difference quotient to the statement that* $\frac{f(x_0+h)-f(x_0-h)}{2h}$ *can work*	C-FACT, created by a PHENOMENIC ACTION guided by a graphic exploration, namely 1) the representation on the x-y axis of $\frac{f(x_0+h)-f(x_0)}{h}$ and its dynamics when $h \to 0$ (already known by the students); 2) the representation of $\frac{f(x_0+h)-f(x_0-h)}{2h}$. The C-FACT is expressed by an UNSTABLE STATEMENT since a sort of analogy between the two graphic representations seem to be not enough to justify the act of reasoning.
$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$ *could represent the slope of the tangent line.* The problem now becomes the point of tangency. A graphic exploration takes place bringing them to state that x_0+h and x_0-h go to x_0 when $h \to 0$, but the result that the numerator of both expressions $f(x_0+h)-f(x_0)$ and $f(x_0+h)-f(x_0-h)$ go to zero when h goes to zero leads them far away from the target (to understand which the point of tangency is); and they don't realize that they considered the dynamics of both denominators from the graph and this ended with the assumption of x_0, while they considered the result of both numerators from an algebraic point of view and this led them to state that it was zero. This situation seems to destabilize Alice and Marco who start an algebraic manipulation in order to obtain, at least in part, the expression of the standard difference quotient.	C-FACT created by a PHENOMENIC ACTION guided by graphic exploration and by the analogy with a dynamics already known. It is expressed by an UNSTABLE STATEMENT since the means used seem not to be sufficient to justify it.

at the end they arrive to separate $\frac{f(x_0+h)-f(x_0)}{h}$ *and* $\frac{f(x_0)-f(x_0-h)}{h}$ *and to state that* $\frac{f(x_0)-f(x_0-h)}{h}$ *is the difference quotient only that there is a minus; namely, it is the same thing but considered at the other side.*	Creation of a FACT generated by a PHENOMENIC ACTION, guided by a graphic exploration. It is expressed by an STABLE STATEMENT, because the graphic exploration seems to be enough to justify the act of reasoning.

5.2 Analysis of a lesson given by the professor

The analysis of the lecture is aimed at examining the structure of the teacher's dialogue, indicating creative abductive processes, and to compare these with the attitudes observed in the students. The analysis follows the same procedure adopted for the protocols: it is divided into two phases; the first phase shows a comprehensive description of teacher's behaviors in tackling the topic, in the second phase the creative processes are detected and interpreted through the elements of the abductive system.

A table divided into three columns represent the structure of the second phase of the analysis; the first column has been used to write the excerpts considered relevant to the creative processes; the second column has been used to write the interpretation through the tools of the abductive system in the teacher's perspective; the third column has been used to write the interpretation through the tools of the abductive system in the student's perspective.

5.2.1 The lecture

The lecture concerns the proposal of some tasks regarding the continuity and differentiability of the one-variable functions. The following analysis proposes only some parts of the lecture, having chosen the most significant parts related to the creative abductive processes.

The initial approach of the teacher consists of stating the definition of differentiability, and it is written on the blackboard; this issue is introduced underlying that the differentiability is the last topic seen by the student in the theory.

$$f: (a,b) \to \boldsymbol{R} \qquad x_0 \in (a,b)$$

$$f \textit{ is differentiable in } x_0$$

$$\textit{if } \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \textit{ exists and belongs to } \boldsymbol{R}$$

The subsequent step is characterized by the geometrical interpretation, and the problem is introduced by a question:

T: *what does it mean from a geometrical point of view?*

At this point he draws a graph

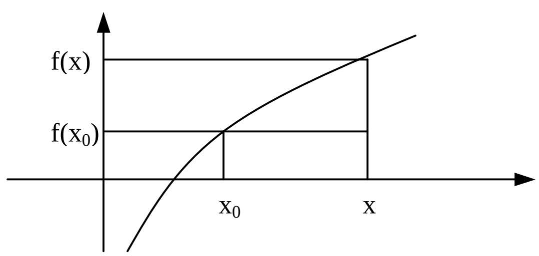


Figure 17: First graph for the geometrical interpretation of the first derivative

The teacher reconsiders the expression $\frac{f(x) - f(x_0)}{x - x_0}$ saying: *this object is named difference quotient*, justifying the term through the graphic and showing that at the numerator there is the increment of f (note: he visualizes it on the graph)

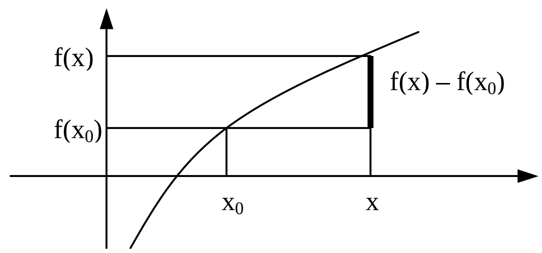


Figure 18: Graphical visualization of the increment f(x)- f(x_0)

While at the denominator the increment of x (note: he visualizes it on the graph, too)

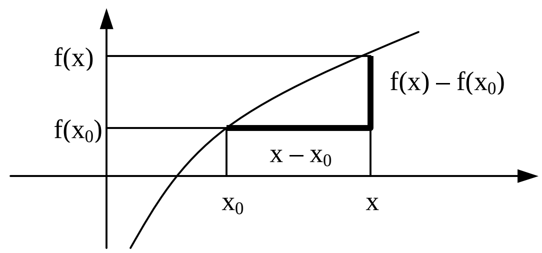


Figure 19: Graphical visualization of the increment x-x_0

The difference quotient, at this point, is interpreted as the slope of a tangent, which is drawn on the graph

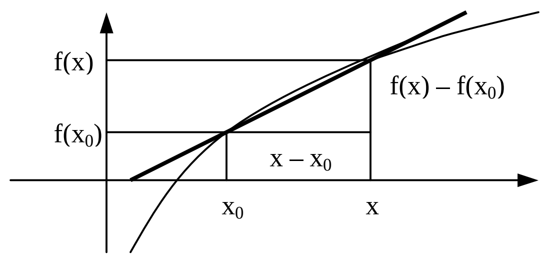


Figure 20: The line through the points (x_0, f(x_0)) and (x, f(x))

In this phase the gestures, the use of the graphs and the reference to them become fundamental tools. The professor refers to the limit "imitating one the graph" the approaching of the line while x approaches x_0 and showing the transformation of the line into tangent line.

Subsequently, he recalls another formula: *Do you remember the formula of the tangent line?* (note: even in this case the introduction of a concept occurs with a question) *(x_0, $f(x_0)$) is*

$y = f(x_0) + f'(x_0)\,(x-x_0)$

this because it is a line that goes through the point (x_0, $f(x_0)$) and its slope is the first derivative.

The basic idea is to link the expression

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

with its geometrical meaning, using the graphical visualization; and the further link between the first derivative and the formal expression of the tangent line.

The teacher, at a certain point, feels the necessity to graphically reinforce the idea of continuity, because for each step there is the intention to give a sense that goes beyond the formalism; to this extent he compares two graphs (see Figure 21 and Figure 22)

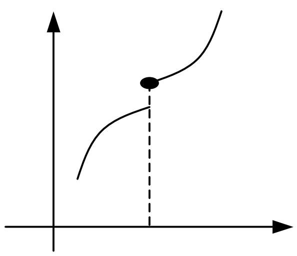


Figure 21: Discontinuous function

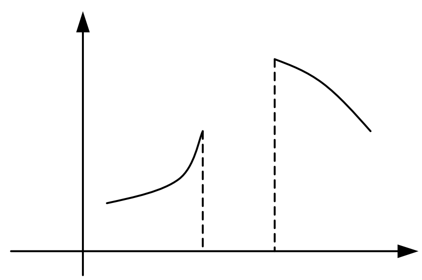


Figure 22: Continuous function

underlying the fact that both present "gaps", but the first represents the graph of a discontinuous function while the second represents the graph of a continuous function, and stressing that the concept of continuity is related to the idea: “for small variations of x we have small variations of y”.

The teacher’s attitude reflects a scheme of abductive type, since it starts with the taking into consideration of a fact followed by the search of possible regularities justifying the observed fact. With the same approach, the teacher compares the following two graphs

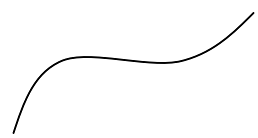


Figure 23: Differentiable function ∀x

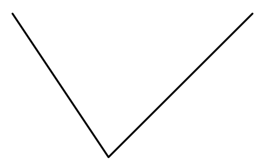
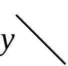
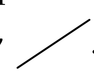


Figure 24: Function differentiable not ∀x

emphasizing the concept that a differentiable function has always a tangent line; while in the case of the Figure 24, in the minimum value the function is not differentiable because in such point it doesn’t have a tangent line, or better, in this point it has a tangent line that *immediately changes the slope from this way*  *to this way*  .

The lecture follows with the question: *How can we say that the first derivative of x^2 is 2x? Don’t give the usual answer: the teacher said so, that’s all.* The proof continues with the application of the definition and further theorems.

Another abductive approach is present when the teacher emphasizes a quite frequent mistake: *Very often when you are asked to find where a function is differentiable, you usually calculate the first derivative and then you study the domain of it, and this set becomes the differentiability set of the function. For example, if you were* *asked to find where the first derivative of lnx is defined, I hope nobody will do the usual thing: since the first derivative of lnx is 1/x, usually people answer* $\forall x \in \mathbf{R} \setminus \{0\}$, because 1/x is its first derivative, defined in $\mathbf{R} \setminus \{0\}$, *be careful!…the first derivative…1/x…let's say…do you remember when we introduced the functions, we said that the function is defined by a law and by a domain , then this is the law (note: referring to 1/x) but the domain is not brought by it, it is difficult to say that the first derivative is defined for negative values, if the function is not defined there, is the problem clear? I mean, this function is defined only for the positive x* (note: he draws the graph of lnx).

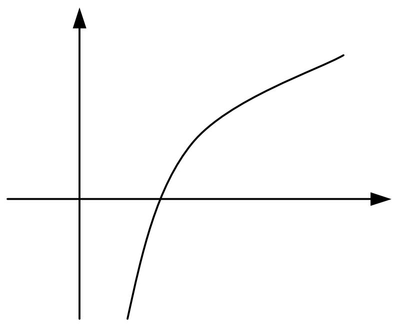


Figure 25: Graph of f(x) = lnx

Form here the teacher continues with further consideration about the domain of lnx.

A subsequent abductive approach has been found with the introduction of the theorem linking the sign of the first derivative and the increasing or decreasing of a function…*Well…a possible consequence of the derivatives, for example, is: if the first derivative is greater or equal to zero then the function is increasing* (note: he writes the following formalization)

f'(x) ≥ 0 ⇒ f is increasing

more precisely f'(x) ≥ 0 ⇔ f is increasing

if the function is differentiable (and he arranges in the following way)

if f is differentiable

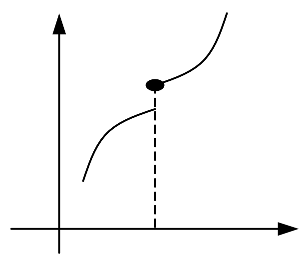
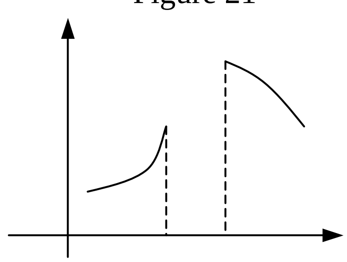
f'(x) ≥ 0 ⇔ f is increasing

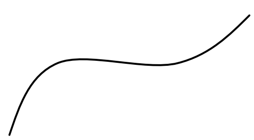
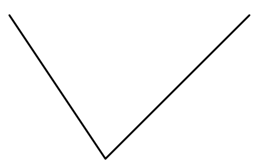
is it clear?

*The theorem is obvious if you have understood what the derivative is: it is the slope of the tangent line (*note: he imitates it with his hands). *If the slope is positive that means that the function is going up* (note: he imitates with his hands) *if the slope of the tangent line is negative then the function goes down* (note: again he shows it with the hands).

The attitude is then an abductive one, since there is the presentation of a fact ($f'(x) \geq 0 \Leftrightarrow$ f is increasing) and then there is the search of possible explanations, in this case the geometrical interpretation of the first derivative as the slope of the tangent line and its relationship with the graph of the function. The lesson continues presenting several approaches of abductive nature.

5.2.2 Analysis through the tools of the Abductive System

Excerpt	**Interpretation through the tools of the Abductive System**	
	For the Teacher	**For the Student**
 Figure 21  Figure 22 *The graph in Figure 21 represents a discontinuous function, the graph in Figure 22 represents a discontinuous function.*	FACT created by a PHENOMENIC ACTION guided by the need to make students understand the concept of continuity beyond the formal definition. It is expressed by a STABLE STATEMENT**,** because the teacher owns the cultural background that justifies such a fact.	C-FACT expressed by an UNSTABLE STATEMENT, because the visual impact should not be enough.

For small variations of x we have small variations of y.	Creation of a HYPOTHESIS through an ABDUCTIVE ACTION guided by the definition of continuous function. The hypothesis is stated by a STABLE ABDUCTIVE STATEMENT, because the definition seems to be enough to legitimate the hypothesis	Creation of a HYPOTHESIS through an ABDUCTIVE ACTION The hypothesis is stated by a STABLE ABDUCTIVE STATEMENT, because the definition seems to be enough to legitimate the hypothesis
 Figure 23  Figure 24 *The function in Figure 23 is differentiable everywhere, the function in the Figure 24 is not differentiable everywhere*	FACT created by a PHENOMENIC ACTION guided by the need to make students understand the concept of differentiability through its graphic meaning. It is expressed by a STABLE STATEMENT**,** because the teacher owns the cultural background that justifies such a fact.	C-FACT expressed by an UNSTABLE STATEMENT, because the visual impact should not be enough.
The function in Figure 24 is not differentiable in the minimum value because in this it doesn't have a tangent line, or better, in this point it has a tangent line that *immediately changes the slope from this way* *to* *this way*	Creation of a HYPOTHESIS through an ABDUCTIVE ACTION guided by the relationship between differentiable functions and the geometrical meaning of the first	Creation of a HYPOTHESIS It is expressed by a STABLE ABDUCTIVE STATEMENT, because the definition seems to be enough to legitimate the hypothesis

	derivative. It is expressed by a STABLE ABDUCTIVE STATEMENT, because the definition seems to be enough to legitimate the hypothesis	
Well…a possible consequence of the derivatives, for example, is: if the first derivative is greater or equal to zero then the function is increasing (note: he writes the following formalization) *$f'(x) \geq 0 \Rightarrow$ f is increasing* *more precisely* *$f'(x) \geq 0 \Leftrightarrow$ f is increasing* *if the function is differentiable* (and he arranges in the following way) *if f is differentiable* *$f'(x) \geq 0 \Leftrightarrow$ f is increasing* *is it clear? The theorem is obvious if you have understood what the derivative is: it is the slope of the tangent line*	FACT created by a PHENOMENIC ACTION guided by the need to show the sense and the need of the first derivative. It is expressed by a STABLE STATEMENT, because the teacher owns the cultural background that justifies such a fact.	C-FACT expressed by an UNSTABLE STATEMENT, since for the student, so far, it is just the statement of a rule (theorem)
The first derivative is the slope of the tangent line. If the slope is positive that means that the function is going up (note: he imitates with his hands) *if the slope of the tangent line is negative then the function goes down* (note: again he shows it with the hands).	Creation of a HYPOTHESIS through an ABDUCTIVE ACTION guided by the relationship between the geometrical meaning of the first derivative and the graph of a function. It is expressed by a STABLE	HYPOTHESIS. It is expressed by a STABLE ABDUCTIVE STATEMENT, because the visualization of the dynamic behaviour of the function seems to be enough to legitimate the hypothesis

	ABDUCTIVE STATEMENT, because the visualization of the dynamic behaviour of the function seems to be enough to legitimate the hypothesis	

5.3 Brief analysis using the reference system continuity

The idea of Continuity as it has been introduced by Garuti, Boero and Mariotti (1998), and redefined by Pedemonte (2002) like Reference System Continuity has made me think if I could use such a definition to look for possible continuities or breaks between the creation process of a c-fact and the creation of the hypothesis justifying the conjectured fact. Such a use of the "continuity tool" differs from its original utilization in the sense that I am not interested in looking for possible breaks or continuities between the conjecturing phase and the proving phase (in the manner intended by the researchers who defined the Cognitive Unity), but in possible breaks or continuities between the phenomenic actions and the abductive actions.. My aim is to understand if the continuity between the tools used in the construction of the c-fact and the construction of the hypothesis may facilitate this last step, or if, at this stage, such a continuity is irrelevant.

The analysis of the protocols has evidenced that the students who successfully achieved a correct solution of the problem, not necessarily have maintained continuity between the phenomenic actions and the abductive actions. The following excerpts are examples of this phenomenon:

<u>Marco and Matteo (fixed point problem)</u>

In the conjecturing phase they build the conjectured fact, *f probably has fixed points,* guided by a didactical contract (as mentioned in the analysis of protocol through the tools of the Abdcutive System), while the hypothesis justifying the c-fact, *the continuous function in [0,1] intersect the bisector line,* has been constructed by an abductive action conveyed by a visual impact. In this case we can talk of break in the

Reference System Continuity, since we have a register based on “contract”, for what concerns the phenomenic action, and a register based on heuristics for what concerns the abductive action.

A different situation is presented by Daniele and Betta, who achieved a correct solution of limit problem but which have evidenced continuity between phenomenic and abducitve actions:

<u>Daniele and Betta (limit problem)</u>

In the conjecturing phase Daniele and Betta state that: *In my opinion it is the same thing…*

Namely, doing $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$ is the same as $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$

The c-fact is created by a visual impact with the graphic representation met for the limit of the standard difference quotient. In the same way, the hypothesis: *The two limits use the same tools*, has been created by an abductive action conveyed by a graphic interpretation. In this case we are in front of continuity between the two stages.

For what concerns the Reference System Continuity (as defined by Garuti & al., 1998; Pedemonte, 2002), the analysis of the protocols has evidenced the presence of such continuity since the means employed by the students in the construction of the conjectures are maintained in the evidencing process; the following excerpt is an example of this phenomenon.

<u>Alice and Roberta (fixed point problem)</u>

<u>R3</u>**: A:** (she draws the bisector line) *therefore this is the (1,1) and (0,0).* The idea becomes sign (Figure 14)

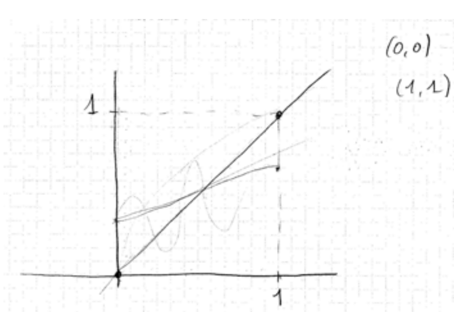


Figure 14

<u>R4</u>**: R:** *The fixed point must be between these two points …*(and she signs the two points (0,0) and (1,1) going along the bisector line)

Consideration of a fact: “the fixed point is on the segment with end points (0,0) and (1,1).” Such fact seems to be legitimated by the property previously considered (namely, the fixed points lay on the bisector line), and by the domain of f which is [0,1].

<u>R5</u>**: A**: *Exactly…but it could have only these two points [(0,0), (1,1)]; if it were in this way* (and she signs a concave function over the bisector line) *therefore there is a fixed point for sure, because there are these two points of the bisector line* (and she signs (0,0) and (1,1))

In this part of the conjecturing phase Alice and Roberta use perceptive, graphic aids. In the evidencing phase it is possible to find the same register, as it has show below.

<u>R27</u>**: A/R:** *then the function must start from 0 and have f(x) on this side and arrive at the point of abscissa x=1 and f(x) on this side then…there is the bisector line that goes through (0,0) and (1,1)*

<u>R33</u>**: A:** (Alice writes) *then* $P(0, 0\leq y \leq 1)$ *because the domain…*

<u>R34</u>**: R:** *it is defined from 0 to 1*

<u>R37</u>**: A** (Alice writes) *it must exist too…*P_1 $(1, 0\leq y \leq 1)$*, I would start with the limit cases,* $P(0,0)$ *and* P_1 $(1,1)$ *or when* (she goes with her finger from the point (0,0) along the segment 0-1 on the y-axes, and she does the same with the punt (1,1) downwards)

At this point they write on their protocol:

If the function f(x) goes through P(0,0), a fixed point is P; There could exist other fixed points in the case that the function intersects the bisector line.

In the same way, if the function goes through the point P(1,1). In all other cases the function will have to go through a point with abscissa 0 and a point of abscissa 1 (for hypothesis). In these cases the ordinate of the point with abscissa 0 will have to be $0 \leq y \leq 1$*, and the ordinate of the point with abscissa 1 will have to be* $0 \leq y \leq 1$*. Being the function continuous for any path satisfying the aforementioned conditions will have to intersect the bisector line in at least one point (on the bisector line lie all the fixed points).*

6. DISCUSSION

The analysis of the protocols has highlighted the presence of creative abductive processes. The importance of these kinds of processes in mathematics, but more generally in the sciences, will be discussed in the first section of this chapter. The existence of abductive processes in the students' works also brings to light the necessity to wonder which elements may promote such a method of reasoning; to this extent the following three sections are dedicated to the analysis of three different conditions, which seem to enhance the manifestation of creative abductive processes. Briefly, these conditions are:

1. A didactical contract that encourages and emphasizes creative processes aimed at understanding *how things work* in mathematics (paragraph 6.2).
2. The chance of favoring (with an appropriate choice of tasks) transformational and perceptual reasoning (Harel, 1998) to pass from the phase of exploration to the phase of creative abductive act of reasoning (paragraph 6.3).
3. The chance of favoring (with an appropriate choice of tasks) the "reference system continuity" between the conjecturing phase and the evidencing phase, as a facilitating condition for the success of the student, and therefore of his or her satisfaction to fulfill the requirement of the task (paragraph 6.4).

Finally, the experimentation has been conducted with a particular sample of students (paragraph 6.5), since it was necessary to create the optimum conditions in order to study the manifestation of abductive processes and the role of the aforementioned factors.

6.1 The role of abduction in sciences

Thomas Huxley, the famous biologist of the second half of the nineteenth century, talked about *retrospective prophecy* to signify the inquiry in the relationship between the cause and effect of a phenomenon, meant as the proceeding backwards, trying to abduce, from what one sees, what may have caused said phenomenon.

Abduction is fundamental in the sciences which study the past: the historian draws the follow-up of the events from the documents and from the proofs that have come down to us; the archeologist goes back to the lifestyles of the ancient populations using what remains of their architectonical structures, or of their utensils; the paleontologist reconstructs the aspect of a prehistoric animal from the fragments of its skeleton and of its teeth, and thinking over these few elements he may decide if the animal is aquatic or terrestrial, if it is carnivorous or herbivorous, and so on.

Huxley asserted that the method of the *retrospective prophecy* is innate in each of us since any daily action is based on the common sense consideration that a certain effect implies a certain cause. But the English scientist went beyond this, claiming that if such method is valid for some sciences, then it has to be valid for all of them.

The Scottish doctor, Joseph Bell, who explicitly referred to the method of the *retrospective prophecy,* argued that the precise and intelligent identification and the taking into consideration of the smallest differences is the real essential factor in all correct diagnosis. On the other hand, as underlined many times in the scientific field, even the sharpest sense of observation, accompanied by memory and imagination, requires, to arrive at the target, a prepared mind from the cultural point of view and a readiness to associate in a coherent manner the available elements. Bell claimed that there are many eloquent and instructive signs, but they require a prepared eye to be identified.

The purpose of this preamble is to underline the fact that an abductive attitude has probably an innate aspect based on common sense, as a natural inclination of the human learner who seeks to understand and to validate an observation; but on the other hand, as Simon (1996) says about transformational reasoning, we could also say about abductive processes, that "this inclination, like many other inclinations (the desire to draw what one sees, to find patterns in one's world of experience) must be nurtured and developed".

It is therefore necessary to stress that the abductive processes met in the analysis of the protocols cannot be related to an inclination of the human nature alone, but they probably depend on the scholastic and extra-scholastic experience of the student. The following sections will consider some of the issues related to this scholastic experience.

6.2 The role of the didactical contract

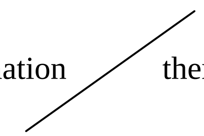
In the specific case of this research there are two didactical contracts to be considered: the first one, between teacher and student, and the second one between student and researcher.

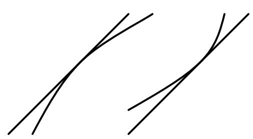
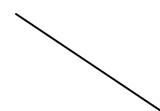
Student and Researcher: For what concerns my role, it is important to underscore that before being the researcher involved in this project, I am the Teaching Assistant of the course of Calculus the students are taking. Throughout the semester, the students and I will share three-hour lesson each week, and furthermore any office hour time they will consider necessary, during which they will be able to ask for further explanations of the exercises, or of the theory behind the lesson.

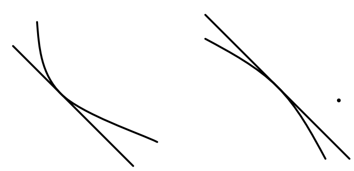
The three-hour lesson is divided in three phases; during the first phase I will suggest some exercises on which the students will work autonomously, alone or in groups; they will freely decide which working groups to form, I will circulate through the classroom intervening only when the students request. From the beginning I emphasize that it is important they try to solve the problem with any means they believe correct; they do not have to be afraid to make mistakes, since they will not be judged for that, what it is important is their desire to understand and to do their best. The chance to call me and to ask me questions personally, gives the opportunity even to the most reticent students who in front of the class would not feel comfortable for fear of being judged for their questions. The first phase usually takes about two hours, as already said I continue to underline the importance to feel free to make mistakes, to ask questions, to exchange ideas with other working groups, to be unafraid to change strategies if they find out they are following an incorrect path, and finally that I will not continue second phase until I can determine that all of them have given the work their best efforts..

In the second phase, I ask students to show their solutions, leaving them completely free to decide if they want to do so or not; in the previous phase I had the opportunity to see the various attempts made by the students, to listen to their difficulties and perplexities; therefore I have been able to ascertain an idea of the important issues I have to touch on if they do not emerge from the solutions proposed by the students. The solutions will be written on the blackboard and together will discuss their correctness; in this phase the formalization will not be considered the most important thing, but emphasis will be given to the correctness of the process. The last phase is used “to make the point of the situation”; the problem is summarized and the correct solutions, previously considered, are rewritten in a more structured way.

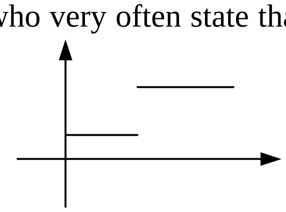
Student and Teacher: after attending some of the teacher’s lectures, and having talked with him, I realized that one of the most important messages he wants to give to his students is that, beyond the formalization, he would like his students to understand how things work, especially from a geometrical point of view. An example given to me by the professor is the approach taken by the theorem linking the sign of the first derivative and the increasing and decreasing of the function. From one side, he uses the geometric interpretation of the first derivative and shows that if the

tangent line has this inclination  therefore the function is necessarily

increasing  and if the line has this inclination 

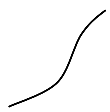
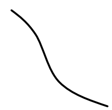
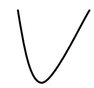
therefore the function is necessarily decreasing  .

After having introduced the formula of Taylor, the professor underlines that from $f(x) = f(x_0) + f'(x_0)(x-x_0) + 1/2\, f''(x_0)(x-x_0)^2 + \ldots$, the expression $f(x_0) + f'(x_0)(x-x_0)$ represents the approximation of the first order of f(x), where $f'(x_0)$ gives the slope upwards or downwards; while the expression $f(x_0) + f'(x_0)(x-x_0) + 1/2\, f''(x_0)(x-x_0)^2$ is the approximation of the second order where $f''(x_0)$ is the coefficient of a parabola and therefore it gives the convexity of the function. In this case the professor told me that he proceeds with a formal proof (one of the few) showing it as a consequence of the Lagrange theorem, because he considers this proof quite simple and because it allows showing a very frequent mistake made by the students who very often state that $f'(x) = 0 \Leftrightarrow f(x)$ is constant.

To this extent, the teacher gives the following example 

underscoring that they incorrectly apply the Lagrange theorem, which is instead applicable only to the function defined in one interval.

The formalization is then applied when is considered quite simple to be understood and when it is useful for the understanding of a further concept. Following this approach, the teacher explains why if $f^n(x_0)$ is the first of the derivatives different to zero and n is odd then the function does not have a maximum or minimum, while if n is even then we have to look for the sign of $f^n(x_0)$. In this case he does not make any formal proof but he recalls again the Taylor formula and asserts that if n is odd then the approximation of the function has

this kind of graph  or  while if n is even the graph is like 

or  and then we will need to look for the sign to understand if we have a maximum or a minimum.

Furthermore, the analysis of the teacher's lecture brought to light what I defined as an *Abductive Scheme*. By means of this definition I want to describe the teacher's attitude adopted in some steps of the didactical transposition, when the teacher wants to convey a creative process, which is already known by him, though. The *Abductive Scheme* has the following structure:

1st step: Proposal of an *act of reasoning*;

2nd step: For the teacher the act of reasoning has value of fact, since he knows a-priori its truthfulness or falseness; the statement expressing the fact is therefore a stable statement. For the student the same act of reasoning becomes a c-fact, expressed therefore by an unstable statement and consequently needing a hypothesis validating or refuting it.

Besides the previously discussed reasons, another aim of the teacher is to avoid an Authoritarian Scheme (Harel, 1998) where the student uses, as validating justification, the assertion "it is true because the teacher said so." The use of the term Abductive Scheme, is necessary in order to distinguish it from the definition of Abductive Process, as it has been defined in this research. The process utilized in the didactical transposition can be defined as a "simulation of a creative process," since the teacher already knows what to build and which hypotheses to use in order to validate or refute the constructed fact.

Finally, it is also important to stress the kind of didactical contract related to the oral exam; the students know that to pass they need to show their understanding of what constitutes the base of a theorem, and they need to show their ability to demonstrate in a constructive way the solution to a problem, rather than repeating a well structured formal proof without demonstrating their understanding of said proof.

Concluding from the analysis of the didactical contract, it is possible to claim that during the first year of the Calculus course, the two fundamental phases we work on are the *conjecturing phase* and the *evidencing phase*[17], while the structuring phase, meant as formal arrangement, is employed when it is considered quite simple to understand or as a tool to facilitate the understanding of a further concept.

The same idea is at the core of my research, and the two exercises given to the group of participants follow this line; in fact the participants were not asked to produce any particular "structured solution," my aims being:

a) To be coherent with the didactical contract.

b) To leave the students completely free to decide their solution process and to autonomously evaluate the acceptability of their solution for the learning community, since many students are not necessarily persuaded by deductive proofs (Martin and Harel, 1989; Chazan 1993).

Concerning this last point, the students have shown their idea about proof as a tool that requires creativity, and with the role of validating a statement, and where hypotheses are means arising a-posteriori with the aim to explain in order to validate or refute a conjectured fact.

From the analysis of the protocols, and that of the didactical contract we can conclude that the creative abductive attitude, met in the students, has probably been

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[17] To explain evidencing phase I refer to Harel's definition of Proof Scheme: "By proving we mean the process employed by an individual to remove or create doubts about the truth of an observation;" and such process includes two sub-processes defined as Ascertaining and Persuading. Ascertaining is the process an individual employs to remove her or his doubts. Persuading is the process an individual employs to remove others' doubts about the truth of an observation.

influenced also by their experiences with a didactical contract that encourages a certain kind of approach, like understanding how things work, the making of connections among mathematical ideas, and creating conjectures and validations of mathematical ideas.

6.3 Perceptual, Transformational Reasoning and abductive process

Through the analysis of the students' work, it has been possible to observe the importance of the Transformational Reasoning[18] and Perceptual Reasoning[19]. Many c-facts or conjectures have been created by actions guided by visual impacts, or transformational reasoning, as shown by the following excerpt:

R7: Matteo: *How can we find this fixed point?*

They try to understand which the fixed points are, and they say:

R8: *a fixed point is here, another one is here…* (see Figure 7) and they arrive at the conclusion that the fixed points lie on the bisector line.

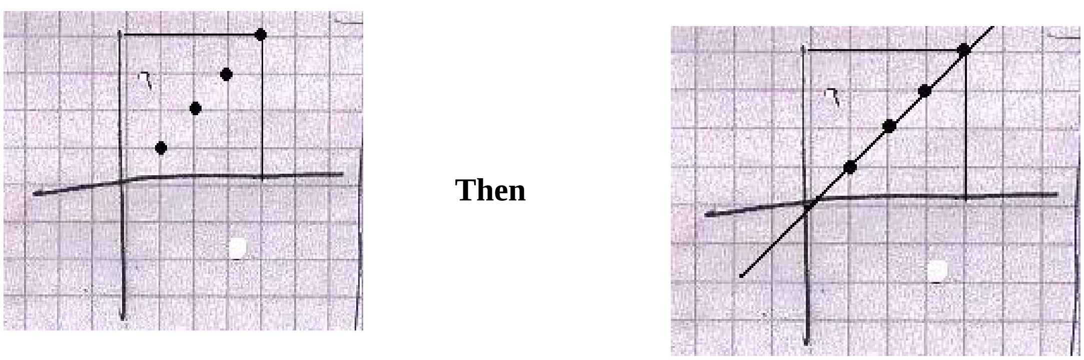

Figure 7. Representation of the fixed points

The construction phase of a possible theory is characterized by a graphic exploration. The graphic aid comes into this: Marco and Matteo, led by the squares on the paper, start identifying the fixed points with ones of the vertexes of the squares, because they satisfy the condition to have the same coordinates, and from the visualization in the

[18] Transformational observations involve operations on objects and anticipations of the operations' results. They are called transformational because they involve transformations of images – perhaps expresses in verbal or written statements – by means of deduction. (p. 258)

[19] The perceptual proof scheme is characterized by perceptual observations made by means of rudimentary mental images – images that consist of perceptions and a coordination of perception, but lack the ability to transform or to anticipate the results of a transformation. (p. 255)

discrete they go to the continuous, hypothesizing that if it is valid on the vertexes you can see, it will be valid for all the "sub-squares" which is made by. Marco and Matteo have the following definition of fixed point: (c, f(c)) with f(c) = c; therefore *c* is the "x" and *f(c)* is the "y"; the subsequent step is represented by their statement that the fixed points are the ones that have "the x equals y" and the y represent it graphically as vertexes of the square of the paper; the idea that the point has the same coordinate allows Marco and Matteo to sign them on the vertexes of the square on the paper. Therefore, the idea is translated in sign, such a sign probably allows a new step, it visually suggests the passage from discrete to continuous…they probably realize, thanks to a visual factor, that between the square represented by the first square of the paper and the second one there are other infinite squares whose vertexes will represent fixed points. Therefore, they draw the line connecting these points; always realizing graphically that what they have just drawn is the bisector line of the I and III orthant and therefore there is a shift to the interpretation of the fixed point represented by the passage from f(c)= c to y = x (Again the sign is a source of thought. A dynamic that goes from outside to inside). There is an identification of the set of the fixed points with the bisector line of the I and III orthant. Therefore in the passage from the discrete to the continuous the graph becomes a source…meant as a new source of thought.

[…]

R17: **Matteo**: *by contradiction we take 'a' that is greater and $\neq 0$ and 'b' minor, now we say by absurd it doesn't go to, at this point 'a' will take in this point here any point in the middle and that $a \neq y$, therefore a point in which y > x always because in a first moment we said that it was greater therefore y must be greater than x and in this other little point here and here and here it will always be greater strictly greater we arrive here where it must be greater than x, at this point we have to take all these points here; its value in 1 cannot be less than 1, equal 1 or more than 1 because it must stay in this interval here, therefore it is absurd.* (Figure 9)

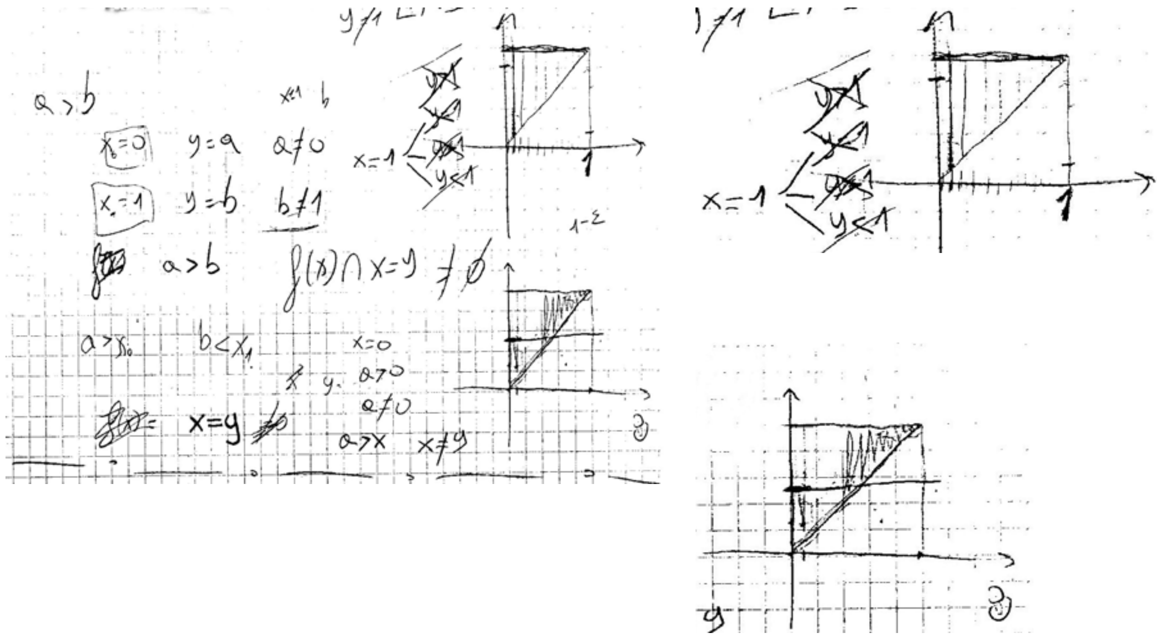

Figure 9. Matteo and Marco’s graphic attempts

In these two excerpts we can observe the importance of the visual impact, meant as the graphic aid employed to build the conjectures, and the transformational reasoning that enhances the evidencing process. In terms of the proof schemes, this work raises a new issue, in the sense that in Harel’s work the problem is tackled mainly from an evidencing point of view while in the abductive system I am proposing here, it takes into consideration the conjecturing face of the process of proving.

6.4 Reference System Continuity and abductive reasoning

The analysis of the protocols has brought to light that it is not possible to relate successful students and continuity between the phenomenic action and abductive action, since it was possible to meet success both in the case of continuity and break.

On the contrary, the analysis of the protocols has highlighted the presence of “reference system continuity” (the one considered by Garuti, Boero & Mariottti, 1996; what has been defined as “cognitive unity of theorems”), while it has not been possible to make an analysis from the point of view of the “structural continuity” (Pedemonte, 2002), since the research was based, as previously mentioned, on the conjecturing and evidencing process and not on the structuring one. Concerning the reference system continuity, the search is done in a narrower process, namely, in the two phases that precede the final step related to the deductive structuring of the proof; the analysis has evidenced the presence of such continuity, as shown by the following example:

In Daniele and Betta's protocol about the exercise on the limit, there are both a graphic and heuristic approaches in the conjecturing phase, which are used in the evidencing phase as well; as evidenced by the following excerpt.

<u>R1</u>: **D**: x_0+h...

<u>R2</u>: **B**: f (x_0)…

<u>R3</u>: **D**: *in my opinion it is the same thing… when you do the limit of the difference quotient, you do* $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ *…this minus this over h…*

He signs on the graph the vertical and the horizontal segments (see the red segments in figure 10)

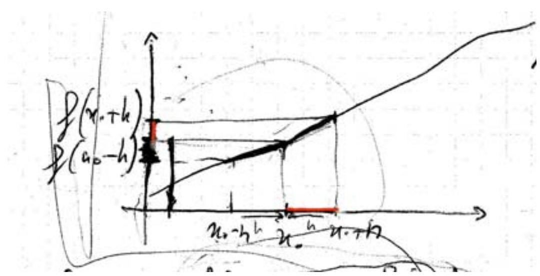

Figure 10. Daniele's graphic interpretation of $\frac{f(x_0+h)-f(x_0)}{h}$

<u>R4</u>: **D**: (note: he signs on the drawing done on the protocol, this $|$ divided by this —)

<u>R5</u>: **B**: *because f(x_0 + h)...*

<u>R6</u>: **D**: *minus f(x_0)...is this*

<u>R7</u>: **B**: *Ah…OK…ours would be this (see the red segments in the figure 11) over 2h…it is the same thing…*

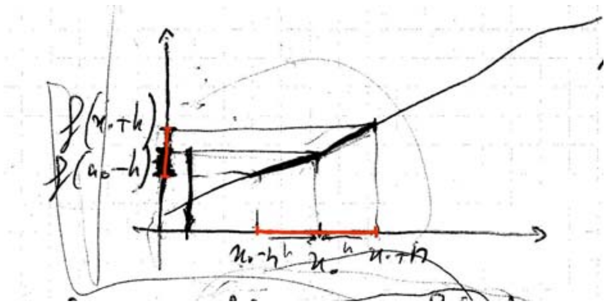

Figure 11. Graphic interpretation of $\dfrac{f(x_0+h)-f(x_0-h)}{2h}$

<u>R8</u>: **D**: *therefore…it would be h→ 0….how much is this?….eh…it will be the slope of the tangent line…*
<u>R9</u>: **B**: *namely…the first derivative*
<u>R10</u>: **D**: *in* x_0

At this point they explain to me their solution to me:
<u>R16</u>: **B:** *this is equal to this* (they indicate the two limits…)…*we done it graphically* (i.e., Betta indicates $\dfrac{f(x_0+h)-f(x_0-h)}{2h}$ and what they have highlighted graphically)
<u>R17</u>: **D**: *I mean, we do this…it would be the ratio between this difference* / *and this one* — *and in our case it would be the ratio between this difference* / *and this one* —, *therefore,* $x_0 + h -(x_0 - h)$ *that would be 2h…and this one that would be* $f(x_0 + h) - f(x_0 - h)$*...therefore, the limit for h that goes to zero would be…I mean both go to* x_0 (note: he shows it to me on the graph).

The central point of the "reference system continuity" lies in the transitional phase from conjecturing to persuading; many times the unity is broken in this passage. The protocols considered in this research have revealed the presence of unity; more precisely, the students' works evidenced that, those who made a correctly reasoned conjecture, then, have arrived successfully at the evidencing phase. Such a phenomenon may be explained by the typology of the proposed exercises, in the sense that the elements used in the conjecturing phase can be used also in the evidencing phase.

6.5 The Sample

The experimentation has been conducted with a particular sample of students, since it was necessary to create the best conditions in order to study the manifestation of abductive processes and the relationships between these processes and the aforementioned factors. The sample taken into consideration, then, is not casual: since the students voluntarily offered to participate in the project, and these students are the students who positively accepted such a challenging situation. Nevertheless, for what concerns the didactical implications, I hypothesize that, since the creative abductive processes don’t seem to be an attitude of a particular elite of subjects (as evidenced in section 1), what has happened with a particular sample of students may be extended to a larger population of students, if the same aforementioned conditions are created.

7 CONCLUSIONS

The *Abductive System* has been created with the aim of providing some tools, which could identify and describe possible creative abductive processes students implement when they perform conjectures and proofs in Calculus. The issue of creativity in the hypothesis creation process led me to consider Charles S. Peirce's work and his definition of Abduction. Subsequently, I realized that the definition of abduction, as given by Peirce, was not sufficient to frame and analyze potential student creative processes, since Peirce's abduction referred to the creation of a hypothesis that could explain an observed fact [20]; while students, very often, are confronted by problems with a direct question, which means the solver not only has to find hypotheses justifying a fact, but also has to look for a fact to be justified. This particularity generated the need to analyze the abductive processes under a new light, in the sense that the nature of the fact and the connections between hypothesis and fact have to be considered in a different way than the manner proposed by a standard abductive process (this relationship will be explained later in the description of the Abductive System).

The construction of my framework has been also influenced by Cifarelli's approach to the concept of abduction. His attention is focused on the abductive inference as a tool to enhance the search for further strategies when the application of a previous solution does not work; therefore the researcher is interested in the role such a process plays on the problem-solving activities.

Reexamining the facts, in Peirce's abduction the fact is a tangible observation: the fossils far in the interior of the country, the white beans on the table; while, for Cifarelli the fact may also be represented by something that happens (see example about Marie's solution, in the core of the research chapter). This last point of view gave me the impetus

[20] In an abductive process a "starting fact" is always considered and it is always true.

to reflect on a new interpretation of the typology of abduction, where the fact is also represented by a strategy/procedure or regularity.

The result of all previous considerations has been the construction of the Abductive System whose elements are {facts, conjectures, statements, actions}.

For *fact* I adopted the definitions of Collins' Dictionary: (1) *referring to something as a* ***fact*** *means to think it is true or correct;* (2) ***facts*** *are pieces of information that can be discovered*.

For *conjectures* I adopted the definition given by Webster's dictionary, ***conjecture*** *is an opinion or judgment, formed on defective or presumptive evidence; probable inference; surmise; guess; suspicion.*

The *conjectures* assume a double role of:

1. *Hypothesis* - an idea that is suggested as a possible explanation for a particular situation or condition.
2. *C-Fact (conjectured-fact)* - final answer to the problem, or answer to certain steps of the solving process.

Statements divided into the three following categories express *Facts* and Conjectures:

1. Stable statements
2. Unstable statements
3. Abductive statements

A *stable statement* is a proposition whose truthfulness and reliability are guaranteed, according to the individual, by the tools used to build or consider the fact or conjecture described by the proposition itself. Namely, the truthfulness depends directly on the tools employed in the construction phase (E.g. a "visually-based" fact: the validity of the proposition describing the phenomenon is justified by a visual perception).

An *unstable statement* is a proposition whose truthfulness and reliability are <u>not</u> guaranteed, according to the individual, by the tools used to build or consider the conjecture described by the proposition itself. Namely, the tools used in the creation phase are not sufficient for the solver to consider the conjecture described by the proposition as being definitively true. The consequence of this is the search of a hypothesis, and / or an argumentation that might validate the aforementioned statement.

An *abductive statement* is a proposition describing a hypothesis built in order to corroborate or to explain a conjecture. The abductive statements too, may also be divided into stable and unstable abductive statements. The former, according to the solver, state hypotheses that do not need further proof; the latter require a proof to be validated.

It is important to clarify that the definitions of *stable*[21] and *unstable statement* are student-centered, namely, the condition of stable and unstable is related to the subject: what can be stable for one student may represent an unstable statement for another student and vice versa; not only that, but the same subject may believe stable a particular statement at a certain point of their scholastic career, and this may become unstable later on when their cultural knowledge base of structured mathematical knowledge increases (e.g.; she or he learns new mathematical systems; new axioms and theorems). Furthermore, a stable statement may become unstable, inside a similar problem-solving process, not because the student is convinced of that, but for a “cultural contract”; namely, the student may recall their scholastic experience and remember that a statement is considered stable if it is justified inside a precise mathematical system supported by axioms, and theorems; thus they will analyze the tools employed for verification if they satisfy such conditions. Another situation leading the student to reconsider a statement from stable to unstable is the “didactical contract”; the subject might believe the visual evidence to be sufficient in order to justify a conjecture, but the intervention of the teacher could underline its insufficiency and therefore the student would find themselves looking for new tools. Furthermore, the same statement may transform from unstable to

[21]The concepts of stable and unstable are related, moreover, to the mathematical context. In Euclidean Geometry if a statement is stable, the problem will be only to find the tools to prove it. Namely, in Euclidean Geometry it is enough to find few variations of “targeted” drawings to guarantee the stability of a statement. In Arithmetic the problem is more complex; it is sufficient to think of Goldbach’s conjecture. Goldabach’s original conjecture (sometimes called the “ternary” Goldbach conjecture), written in 1742 in a letter to Euler, states “at least it seems that every number that is greater than 2 is the sum of three primes”. Note that here Goldbach considered the number 1 to be prime, a convention that is no longer followed. As re-expressed by Euler, an equivalent form of this conjecture (called the “strong” or “binary” Goldbach conjecture) asserts that all positive even integers ≥ 4 can be expressed as the sum of two primes. Not only a proof has not been found yet, but also, even though many millions of even numbers have satisfied such property, we are still not sure of its validity.

stable inside a similar process because the subject follows the mathematician's path: they starts *browsing* just to look for any idea in order to become sufficiently convinced of the truth of their observation, then they turn to the *formal-theoretical world* in order to give to their idea a character of reliability for all the community (Thurston, 1994).

Behind any statement there is an action. *Actions* are divided into *phenomenic* actions and *abductive* actions. A *phenomenic action* represents the creation, or the "taking into consideration" of a fact or a c-fact: such a process may use any kind of tools; for example, visual analogies evoking already observed facts, a simple guess, or a feeling, "that it could be in that way;" a phenomenic action may be guided, for example, by a didactical contract or by a transformational reasoning (Harel, 1998). An *abductive action* represents the creation, or the "taking into account" of a justifying hypothesis or a cause; like the phenomenic action, they may be conveyed by a process of interiorization (Harel, 1998), by transformational reasoning (Harel, 1998) and so on.

The Abductive System could be schematized in the following way: *conjectures* and *facts* are 'act[s]} of reasoning' (Boero 1995) generated by phenomenic or abductive actions, and expressed by 'act[s] of speech' (ibid) which are the statements. The adjectives *stable*, *unstable,* and *abductive* are not related to the words of the statements but to the acts of reasoning of which they are the expression. Hence, the only tangible thing is the act of speech, but from there we may go back to a judgment concerning the act of reasoning thanks to the adjectives given to the statement. Finally, for two different subjects the same statement may be stable or unstable. Therefore, two people may achieve the same act of reasoning and judge it by a different method.

At the base of the construction of the Abductive System there is also the intention to show that the creative processes own some components, and to separate these processes from the belief that it is not possible to talk about it because it is something indefinable and only comparable to a "flash of genius". The common denominator with Peirce's work is the philosophic spirit on which both works are based. Peirce wanted to legitimate the fact that abduction is a kind of reasoning along with deduction and induction, in contrast with many philosophers who regard the discovery of new ideas as mere guesswork, chance, insight, hunch or some mental jump of the scientist that is only open to historical, psychological, or sociological investigation.

The research questions leading my work are:

1. Are the definitions of abduction, already given, sufficient to describe creative processes of an abductive nature? Or, is a broader definition of abductive process needed to understand some creative students' processes in mathematics proving? If so, what is that definition?
2. Is one's certainty about the truth of an assumption an indication of an initiation of abductive reasoning in her or his process? Namely, how important is the level of confidence of the constructed answer in guiding an abductive approach?
3. Is there continuity between the cognitive "tools" one uses to build a conjecture and the means one uses to establish its validity?
4. Which elements convey an abductive process? In particular, does transformational reasoning facilitate an abductive process?

The definition of Abductive System allows the researcher to analyze a broader spectrum of creative processes than those covered by the already given definitions of abduction, and the experimental phase revealed to show the presence of those components I have given a name inside the Abductive System.

The analysis of the data through the tools of the Abductive System allowed answering to the previous questions. Indeed the Abductive System, in general, is a possible answer to the first question, having broadened the definition of abduction and the distinction between stable and unstable statements probably guide the way to a possible conclusion for the second question. When an act of reasoning is expressed by an unstable statement, the subject needs to find a hypothesis that could validate or confute it.

Regarding the third question, it has been possible to find the presence of "reference system continuity" (Garuti, Boero & Mariotti, 1996; what has been defined as "cognitive unity of the theorems"), but it has not been possible to make an analysis from the "structural continuity" (Pedemonte, 2002) point of view, since the research has been based on the conjecturing and evidencing process and not on the structuring phase, meant as that process of deductive arrangement; in fact, the participants in the research were not asked to produce any particular “structured” solution; my aim being to leave the students completely free to decide their solution process, and to autonomously evaluate the acceptability of their solution for the learning community.

It is important to underline that “the reference system continuity” has probably been favored by the kind of the problems proposed to the students, in the sense that the elements used in the conjecturing phase could be used in the evidencing phase, as well.

The last question brings to light the issue of the role of the transformational reasoning (as defined by Harel, 1998) in facilitating a possible abductive process; the research has confirmed that perceptual and transformational reasoning have played a fundamental role in the construction of both conjectures (c-facts and hypotheses) and facts.

There is a further factor we need to take into consideration, which is the typology of the sample; it cannot be defined as a casual sample, since the students voluntarily offered to participate in the project, and probably were those students who positively accepted a didactical contract that encourages an approach promoting the understanding how things works, the making of connections among mathematical ideas, creating conjectures and validations of mathematical ideas, rather than a formal deductive approach. Nevertheless, regarding what concerns the didactical implications, I hypothesize that, since the creative abductive processes don’t seem to be an attitude of a particular elite of subjects, what has happened with a particular sample of students may be extended to a larger population of students, if the same previously mentioned conditions are created.

Furthermore, the creative abductive attitude met in the students, cannot be considered only an inclination of human nature, but it also probably depends on the scholastic and extra-scholastic experience of the student, and certain kinds of didactical contract (like those discussed in this work) may positively influence such creative processes.

7.1 Educational implications

The Abductive System may be considered from two perspectives. From the cognitive point of view, it gives the researcher the tools (i.e.; the elements of the Abductive System) to recognize students' creative attitudes during their problem - solving processes. From a didactical point of view, it points to those teaching styles which enhance an "abductive atmosphere" (e.g.; the lecture analyzed in this research), when the teacher does not just deliver the knowledge but he or she creates those conditions where the immediate creation of a fact entails "the necessity" to build or to look for a justifying hypothesis, generating in this way creative mechanisms.

Secondly, the analysis employing the tools of the Abductive System has brought to light the importance of the proposal of "open problems" where the continuity of the cognitive tools and the involvement of transformational and perceptual reasoning are guaranteed, since they seem to improve creative processes of an abductive nature.
Therefore, this framework could help teachers to be more conscious of what has to be 1) recognized, 2) respected, and 3) improved upon, with respect to a didactic culture of "certainty," which follows preestablished schemes.

In terms of Proof Schemes the Abductive System could open a new chapter of the schemes, reconsidering them from the conjecturing point of view, not only from the evidencing one. For what concerns further research issues, it would be interesting to consider two different random groups of students. One group would attend a Calculus course based on a didactical contract similar to that analyzed in this research, and the other group would attend a more traditional Calculus course, where the frontal lecture with a deductive approach is preferred. With the same procedure followed in this research, the two groups would be given some problems to be solved, and their works would be analyzed through the tools of the Abductive System. The focus of the new investigation would be represented by the study of the differences in the solving processes between the two groups, and how different kinds of didactical contract may influence creative processes in the construction of conjectures and proofs.

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APPENDICES

Appendix A: Questionnaire

1st question: CHECK THE FORMS OF REASONING YOU KNOW	
	1st year of college (out of 89)
Induction	9
Deduction	9
Induction, and Deduction	45
Induction, Deduction, and for Contradiction	10
Induction, and for Contradiction	1
Induction, Deduction, and Intuition	3
Induction, and Intuition	1
Induction, Deduction, and Philosophy	1
Induction, Deduction, for Contradiction, and Logic	1
Induction, Deduction, the Tossing of a coin	1
Deduction , and Intuition	1
At least Induction, and deduction	61
Only deduction	9
Only induction	9
For contradiction	12
For intuition	5

COMMENTS

The majority of students (52%) knows both Induction and Deduction; followed by students who know not only Induction and Deduction but also proof by contradiction.

2nd question: AS STUDENT, DO YOU THINK THE STUDY OF PROOFS TO BE NECESSARY?		
		1st year of college (out of 89)
TYPE OF JUSTIFICATION	STUDENTS' TRANSCRIPTS	NUMB OF STUDENTS
Yes	They help to understand theorems and their meaning	17
	They make the content clearer and easier to remember	4
	They explain certain assumptions, and why certain facts happen	3
	They validate the problem, they convince of its validity	7
	They help to create mental schemes and they make you to use them correctly	4
	Because you learn a way to think absolutely connected with the study of mathematics	1
	As a tool to learn to think	1
	As a tool useful to solve problems	6
	As a chain-mechanism: "with one you can learn all of them"	1
	As a tool to deepen	1
	They help to understand better the reasoning used to arrive to the given conclusion	12
	As a generic example to understand the particular rule (for example Rolle)	1
		58
No		
	Because they are difficult and they are just useful for themselves: "They are difficult to understand and there is no interest to understand how a certain thesis has been proven". "It is useful just to understand a formula and that's all"	2
	They are not necessary to solve the problems	1
	Useful only for mathematicians	1

	Because it is a mnemonic study	1
		5
Sometimes		
	When they are useful to understand better the theorems	11
	When the reasoning process is not immediate	3
	When it represents a tool for reasoning	1
	When it is not just useful for itself: “When the concepts are fundamental to understand Calculus/Analysis”. “When the statement to be proved is not just useful for itself but it would result as a tool for future results” “When the theorem is important” “When they are essential for the learning and the good result of a test”	5
	As a tool of exploration	1
	When they do not make the tings more complicate. When They are too complicated they don’t have any didactical value	4
	Only when they really help to understand an issue	1
		26

COMMENTS

58/89 answered **Yes**, 5/89 answered **No**, 26/89 answered **Sometimes**.

Most of the students (65%) think of proofs as a tool to better understand theorems, their meaning, and the reasoning involved into the process of proving. The remaining part is mainly concerned with the idea that proofs are necessary because they validate the problem and convince of its validity, or as a tool useful to solve problems, to create mental schemes to be used in problem-solving, furthermore they explain the why of a fact, and finally they make a context clearer, and easier to be remembered.

Most of the students who answered “Sometimes” (29%) states that proof is necessary when it helps to better understand a theorem.

Very few (6%) are convinced that proofs are not necessary at all.

Conclusion: the main idea about the necessity of a proof is based on its usefulness to help oneself to understand better a theorem; therefore proof is seen as an *explanatory* tool.

3rd question WHICH KIND OF RELATIONSHIP LIES BETWEEN HYPOTHESIS AND THESIS IN THE CONSTRUCTION OF THE STATEMENT OF A THEOREM?		
		1st year of College (out of 89)
TYPE OF JUSTIFICATION	STUDENTS' TRANSCRIPTS	N. OF STUDENTS
It depends		
The thesis is considered a starting point from which you build the hypotheses to prove the validity of the thesis itself.	Many times you start from the thesis and then you build the hypothesis useful to prove the validity of the thesis itself	1
	Sometimes it happens that you have an intuition on a thesis and subsequently you build the hypotheses that make the thesis true; sometimes you start from a set of hypotheses to arrive at some results.	1
	Sometimes you know the thesis and the hypotheses serve to prove the validity of the thesis; other times from the hypotheses you infer the thesis.	1
	Usually you know first the hypotheses to reach the thesis, but you may have also a thesis to be reached and you need to find the hypotheses to make the starting point valid.	1
	Usually the hypothesis comes before the thesis, but sometimes it may be necessary to look for which hypotheses may satisfy a particular thesis	1
	Sometimes the thesis is already known and the proof is used only to explain the why of the validity of the thesis. Other times starting from the hypothesis you reach the proof of the theorem that was previously unknown.	1

	On the situations. Usually you start from an observation; then you see (for example) that certain numbers behave with certain properties and you can build the hypotheses that determine such properties; but it also can happen the opposite. I think it is more common, in the science, to start from the thesis (for example; in physics you first observe the phenomenon); but nothing prevents the opposite process to happen.	1
	Sometimes you know the thesis and hypothesis is necessary to prove that the thesis is valid; sometimes from the hypothesis you deduce the thesis arriving at the concept	1
The awareness of the difference between the construction of a proof and its 'transcription'	I think that there is a difference between the moment you state a theorem (usually the hypotheses are listed in an orderly way, then the thesis go after) and the construction of the statement of a theorem. This one follows a very laborious and "untidy" process; to this extent, sometimes you may have in your mind a result and you need to look for hypotheses from which you obtain the result; other times you start from certain hypotheses and you try to understand what they lead to. Besides, in the famous "if and only if" hypotheses and theses exchange the role.	1
The certainty of the observed fact and the difficulty of the construction of the hypotheses needed to explain it	Many times you know where you want to arrive, but you don't know where to start.	2

Depending on the cultural background and the kind of reasoning	There doesn't exist a precise law, it depends on the kind of research one does and on the cultural background one owns.	1
	On the kind of reasoning. If it is inductive the hypothesis comes first, otherwise the thesis is the one coming first.	3
	On the method that has been used. If deductive the hypothesis comes before the thesis, if inductive the thesis comes before the hypothesis	1
	It depends on the system of reasoning. Example: for contradiction I suppose a thesis to be true in order to verify the validity of the hypothesis	1
The generality is given by 'from hypothesis to thesis', the particular is 'from thesis to hypothesis'	If you are looking for something in particular, I think the thesis comes always before the hypothesis. In the case you want to build, to expand or to deepen a theorem, I think the hypothesis comes first.	1
The idea that the thesis has a more empirical connotation in the sense that it comes from an observed fact. Instead hypothesis has a more cognitive connotation, because it is the construction of an act reasoning	It depends. A theorem often rises from an empirical experience, and therefore the thesis rises before the hypothesis. But it is also true that other times a theorem is the result of a reasoning that starts from very precise hypotheses to reach the theses which can be surprising and unexpected.	1
	Generally the thesis comes before the hypothesis (you try to proof something that will be useful for other purposes), but it can happen that from particular hypotheses a new correct thesis, casually, may rise.	1

The distinction between the two cases: when from a set of hypotheses you infer a thesis, and when given a fact (thesis) you look for some plausible hypotheses to infer the thesis from.	It depends if you infer the thesis from a group of hypothesis (a trivial and not very useful case) or if you need a thesis, or if you want to verify it, and you look for the hypotheses which you infer the thesis from (much more common case).	2
	It depends on where you want to start from. If you suppose the existence of a theorem or if you suppose some conditions in order to arrive to a theorem that you ignore the existence of.	1
	It depends if you have to prove the theorem, or if you have to find it.	3
There is not a particular relationship between hypothesis and thesis	You can nor start a-priori conceptually from a hypothesis and immediately to analyze the thesis; neither can you lead the reasoning starting from the thesis to reach the hypothesis.	1
	In my opinion there isn't a fixed relationship between thesis and hypothesis, and this is proved above all by those theorems in which you can exchange hypothesis and thesis	1
	Sometimes you start from the hypotheses to reach the thesis and sometimes you start from the thesis to reach the hypotheses	3
	It depends because for example for what concerns physics, the phenomena to be studied usually are the thesis, it depends on the scientist to prove how it may happen formulating some hypotheses. On the other hand, sometimes you think if specific hypotheses lead or not to a thesis, and from there, through mathematical steps you arrive at the thesis	1
		32

The thesis always comes before the hypothesis		
The hypothesis as a tool to justify, explain the hypothesis	In the thesis you must hypothesize conditions that make the thesis be verified.	1
	The thesis is the starting point, the “question” that rises after having observed a phenomenon. Instead the hypotheses serve to prove if the thesis is valid or not	1
	Some theorems to be proved need the thesis be denied, therefore the thesis is used as a hypothesis to verify the truthfulness	1
	To proof a thesis you make a hypothesis and then you look if it is true	1
The thesis as a fact, the problem to be solved. There is a sequence between thesis and hypothesis. The presence of a thesis is the necessary condition to have a hypothesis.	Because the thesis is the “problem” you have to solve, the hypotheses are made (imposed) to arrive at the solution	2
	First you “find” a property or a result that if it were valid it would be favorable. Then, you try to proof it under opportune hypotheses. Subsequently you may try to reduce the number of the hypotheses and check if the theorem is still true.	1
	First I decide what has to be proved	1
	Anybody, before of choosing to use particular tools and conditions (hypotheses) to prove his/her own conviction, has to have first a conviction that his/her own genius judges to be correct	1
	The hypotheses of a proof are built afterwards, because they put some “limits” (they are “characteristics”) for the statement of the theorem	1

	Because the statement represents what you wanted to prove, therefore the thesis	1
	Because the aim is always to prove the validity of a thesis, checking among the hypotheses and the data I have the ones which allow me to do so	1
		12
The hypothesis always comes before the thesis		
From known to unknown. Therefore you start from something you already know to prove something that is not still certain. The hypothesis is something already known and from there you start to prove the thesis. But what is the thesis? Is it already known but it is not certain and through the proof and the known hypothesis you arrive to the validity of the thesis? Or do you find out the thesis through the proof? The question is what do we prove if we don't know what to prove?	The hypothesis is already known, the thesis has to be proved starting from the hypothesis	1
	From known things you prove unknown things	1
The hypothesis is the "place" where there are the data you know. Namely, the hypothesis is the "owner" of the data you need for the proof.	In the hypothesis there are the data we know about	2
	Because from initial conditions it is possible to reach a final thesis, on the contrary not always is it possible to reach some initial conditions starting from the final thesis	1

The necessity of the hypothesis to prove the thesis. Probably rose from the way proofs are presented at school.	You always need first a hypothesis	1
	You need to start from the hypothesis to prove the thesis	1
	Without hypotheses you can't arrive at any thesis	1
	On the basis of my hypothesis I state my thesis	1
	The thesis needs an a-priori hypothesis	3
	If there weren't a hypothesis, why would you state a thesis?	1
	Without the hypothesis you can't reach the thesis	3
	Without an hypothesis you can't have a thesis	1
	The hypothesis gives the basis in order to prove the thesis and for its proof, therefore it is essential	1
	Because to prove a thesis you use a hypothesis	1
	To prove a thesis you need a hypothesis	1
This is always a case of hypothesis as an essential tool to make a proof. But there is something more compared with the previous answers: it is explained the reason why the hypothesis is important. It represents the basis to reason about in order to check the validity or falseness of the thesis.	Because the hypothesis represents the data, the bricks on which it is possible to reason in order to find out the truth or the falseness of a thesis	1

Here we can see the description of the structure of a proof as it is shown at school, and not of its creation. According to these answers a proof is a sequence of logical implications which go from hypotheses to thesis, but nothing is said about its creation, just only its structure as a finished product.	From the hypothesis or hypotheses applying mental-logical steps or theorems already known and proved, or axioms, you arrive always at the thesis no matter complicated the theorem is.	1
	Because it is a logical implication (inference)	2
	The thesis is the consequence of a proof that is based on some initial data	1
Again there is the idea of the structure of a proof as a sequence of steps, that start from hypotheses to end into a thesis. Such a rigid structure is so predominant the student doesn't realize that himself assumes the presence of a thesis before the statement of the hypotheses ("the supposed thesis"). Nevertheless, a presence of a thesis before the hypotheses seems not to be part of the process of proving. It also looks like hypotheses live of their own life; the thesis rises as a consequence	Because first you state some hypotheses and then you try to reach the supposed thesis	1
	The thesis is the consequence of a proof that is based on some initial data	1
	You make a hypothesis and then you verify it with experiments to see if they are true. Only then you make the thesis (It is Galileo's model)	1
	Because the thesis arises from the work I do on what the hypothesis says	1
	Because you always make a hypothesis first and then after several proofs you may give a thesis	1
	Given some statements and particular conditions, particular situations follow	1

of the reasoning made about the hypotheses. But why such hypotheses are made or taken on consideration we don't know… Therefore, hypothesis is a necessary condition for the existence of a thesis, due to the fact that this one rises as a consequence of the reasoning made on the hypothesis, but why do such hypotheses come out, pushed by what?	First I state the hypothesis and from that I reason to state my thesis	1
Hypothesis as a start point	Because the hypothesis is a base to start from	1
It seems that the logical sequence is "first doubt and then certainty". To this extent the hypothesis represents the doubt and the thesis is the certainty, therefore the logical sequence of the two.	The hypothesis represents a doubt and the thesis is its confirmation	1

In this case the term "supposition" and "hypothesis" have the same meaning. It seems that the relationship between hypothesis and thesis is the following: the thesis is just a supposition (the hypothesis) we verify the validity of. Namely, I make a supposition (the hypothesis), then I verify if it is true; if it is like that, such hypothesis becomes the thesis. For example: we suppose that the set of the natural number is lower bounded (this is the hypothesis, the supposition), then we prove that it is true, therefore the hypothesis being true, becomes the thesis. Hypothesis and thesis are the same statement with two different value of truth: till when the statement is not proved to be true, it is a hypothesis, after its proof of true value it becomes a thesis.	You make some hypotheses and then you verify with experiments if they are validated, only then you state the thesis	1
	Because I suppose a fact and then I prove that it is true	1
In this specific case there is a clear example of hypothesis meant as a supposition that has to be proved in order to become the thesis. What I observed in these last answers is that students think of hypothesis just as a conjecture to be proved; on the contrary hypotheses meant as a set of rules, axioms etc…already true, are not considered as hypotheses but just a set of statements	Because you suppose a hypothesis to be true, and through a set of statements, you reach a thesis	1
	Because first I suppose some data and I try to verify if my hypotheses are valid or not, if they are not valid I build new hypotheses. Sometimes, anyway, it can happen to discover some formulas or rules, then you try to reach their hypotheses	1
	First I have to develop a hypothesis considering all the elements that have been given	2

but just a set of statements.	First of all you need to have in your mind what you want to prove; in case, after the proof you make adequate changes	1
The thesis as an input to find new things.	Because the thesis has to be an input to look for new properties, new relationships that are based on given elements (the hypotheses)	1
The hypothesis comes first just because its role is to simplify the proof.	Usually the hypothesis simplifies the statement and the proof of the statement, and I believe it is more logical to start from easy and more understandable things to arrive to analyze more complicate ones	3
		45

COMMENTS

32/89 answered **It depends**, 12/89 answered **The thesis comes always before the hypothesis**, 45/89 answered **The hypothesis comes always before the thesis.**

Concerning the first choice (it depends) we could summarize the main justifications as follow: very often the thesis is considered as a starting point from which it is possible to build the hypotheses that may prove the validity of the thesis itself; not only but the thesis seems to own an empirical connotation in contraposition with a more cognitive connotation of the hypothesis; namely, the thesis comes from an observed fact, while hypothesis is the construction of a reasoning.

Very interesting is the answer given by a student who reveal the awareness of the difference between the construction of a proof and its "formalization". Moreover, part of the students relate the characteristics of a proof with the cultural background.

Among the 32 students (36%) who answer "It depends", almost half of them (15/32) seems to base their response "the thesis comes before the hypothesis" on a common idea: the experimental characteristic of the reality; that means: in the real world what is observed is a fact (the thesis) that may be unusual or at least not directly explainable, therefore we look for or we try to build some hypothesis which may justify, or validate the observation that has been made. Such students seem to describe the process Peirce talks about regarding *abduction*. Below the most significant answers given by the students to this regard are listed.

- ✓ *Many times you start from the thesis and then you build the hypothesis useful to prove the validity of the thesis itself*
- ✓ *I think that there is a difference between the moment you state a theorem (usually the hypotheses are listed in an orderly way, then the thesis go after) and the construction of the statement of a theorem. This one follows a very laborious and "untidy" process; to this extent, sometimes you may have in your mind a result and you need to look for hypotheses from which you obtain the result; other times you start from certain hypotheses and you try to understand what they lead to. Besides, in the famous "if and only if" hypotheses and theses exchange the role*.
- ✓ *Many times you know where you want to arrive, but you don't know where to start from.*
- ✓ *Sometimes it happens that you have an intuition on a thesis and subsequently you build the hypotheses that make the thesis true.*
- ✓ *Sometimes the thesis is already known and the proof is used only to explain the why of the validity of the thesis.*
- ✓ *It depends, because often a theorem rises from empirical experience, and therefore the thesis comes before the hypothesis.*
- ✓ *It depends if you infer the thesis from a group of hypothesis (a trivial and not very useful case) or if you need a thesis, or if you want to verify it, and you look for the hypotheses which you infer the thesis from (much more common case).*

On the other hand, the same students who answered "It depends" and gave the explanations listed above, stated that other times "hypothesis comes before thesis"; my feeling on this second kind of response is that has been leaded by the *scholarization* of their vision of proof. Namely, when students enter into school they usually start to approach ready made proofs, well stated, organized as a sequence of logical steps linked one to the other one by deductive processes. All the intuitive, conjecturing phase has been already deleted, and forgotten. Below are reproduced some of the most significant excerpts :

- ✓ *[…] But it is also true that other times a theorem is the result of a reasoning that starts from very precise hypotheses to reach the theses which can be surprising and unexpected.*
- ✓ *[…] Other times starting from the hypothesis you reach the proof of the theorem that was previously unknown.*
- ✓ *[…] Sometimes from the hypothesis you deduce the thesis arriving at the concept.*

Furthermore, I got the impression that students when try to explain why thesis comes before hypothesis they seem really embedded in the reasons they give, otherwise it seems to me that when they try to justify why sometimes the hypothesis comes before thesis they just try to reproduce a frame they have seen at school

The second choice is given by "the thesis comes always before the hypothesis". Again, the general idea supporting this answer is that the thesis is the fact, the problem to be solved, the starting point, and the hypothesis is the tool to explain, to validate the observed fact. A new idea seems to come out from students' justifications, it is the sequence between the hypothesis and the thesis. Namely, the existence of a hypothesis is subordinate to the presence of a thesis as some students wrote:

- ✓ *First I decide what has to be proved*
- ✓ *Anybody, before of choosing to use particular tools and conditions (hypotheses) to prove his/her own conviction, has to have first a conviction that his/her own genius judges to be correct*
- ✓ *The hypotheses of a proof are built afterwards, because they put some "limits" (they are "characteristics") for the statement of the theorem.*

The last choice was represented by: "*the hypothesis comes always before the thesis*". In this case hypothesis, for example, is considered like what you already know and thesis is the unknown, therefore we start from what we know to prove the thesis.

Another interpretation is given by "the hypothesis is the place where the data necessary for the proof lay"; other times the relationship between hypothesis and thesis seems to be the same of the logical sequence "first doubt and then certainty". To this extent the hypothesis represents the doubt and the thesis is the certainty.

The majority of the students seem to be influenced by the structure (and not by the creation) of a proof as it is usually presented at school; therefore, proof is just a sequence of steps, that start from hypotheses to end into a thesis. Such a rigid structure is so predominant the student doesn't realize that he assumes the presence of a thesis before the statement of the hypotheses ("the supposed thesis"). Nevertheless, a presence of a thesis before the hypotheses seems not to be part of the process of proving.

Furthermore, hypotheses seem to live of their own life; the thesis rises as a consequence of the reasoning made about the hypotheses. But why such hypotheses are made or taken on consideration we don't know…

The hypothesis is a necessary tool to make a proof, specially because it is the base to reason about in order to check the validity or falseness of the thesis.

The last interpretation of hypothesis I am going to take on consideration is the most interesting. Several students identify *hypothesis* only with *supposition*, *conjecture*, and look at the thesis as a hypothesis whose true value has been proved; namely, a thesis is a previous hypothesis (conjecture) that has been proved to be true. Therefore, hypothesis and thesis are the same statement with two different value of truth: till when the statement is not proved to be true, it is a hypothesis, after its proof of true value it becomes a thesis. On the contrary hypotheses meant as a set of rules, axioms etc…already true, are not considered as hypotheses but just a set of statements.

Below the most significant excerpts has been taken on consideration to underline the explanations given by the students.

- ✓ *From known things you prove unknown things*
- ✓ *Without hypotheses you can't arrive at any thesis*
- ✓ *The hypothesis gives the basis in order to prove the thesis and for its proof, therefore it is essential*

- ✓ *From the hypothesis or hypotheses applying mental-logical steps or theorems already known and proved, or axioms, you arrive always at the thesis no matter complicated the theorem is.*
- ✓ *Because first you state some hypotheses and then you try to reach the supposed thesis*
- ✓ *Because you always make a hypothesis first and then after several proofs you may give a thesis*
- ✓ *First I state the hypothesis and from that I reason to state my thesis*
- ✓ *Because I suppose a fact and then I prove that it is true*
- ✓ *Because you suppose a hypothesis to be true, and through a set of statements, you reach a thesis*

4th question FOR EACH THEOREM DO YOU THINK THAT THERE EXISTS ONLY ONE CORRECT PROOF?		
		First year of college (out of 89)
TYPE OF JUSTIFICATION	STUDENTS' TRANSCRIPTS	N. OF STUDENTS
Yes. Why?		
	Because I imagine it	1
	Not to create confusion in a proof	1
		2
No. Why?		
It depends on the theorems. Probably such an answer depends on student's scholastic experience	It depends on the theorems, some may have more than one	9
	I know some theorems that have two proofs	3
	It depends on the theorems you are analyzing	1
	I know theorems with more than one proof, for example the Pythagorean theorem.	1

	Many times during high school I saw theorems proved in different ways but all correct	1
	In my opinion there exist more than one correct proof. You may think of "proof by contradiction" that are another way than a "linear" proof	1
Different kinds of reasoning and different tools (e.g., axioms, postulates and so on) lead to different correct proofs. And also different paths you may choose to reach the same target. Furthermore, we can find a sort of "economy" of the thought; we generally choose Different kinds of reasoning and different tools (e.g., axioms, postulates and so on) lead to different correct proofs. And also different paths you may choose to reach the same target.	There are different ways of reasoning, and also different sets of axioms, for example the one of Euclid for algebra and geometry	6
	There exist different ways to reach the same result	20
	It depends on the ways you want to use to reach the proof; many times there are several ways and you always try to choose the most convenient	1

Furthermore, we can find a sort of "economy" of the thought; we generally choose the easiest, or most convenient procedure. Finally, the dependence of the kind of procedure on the knowledge and the cultural background of the person who performs the proof The different levels of knowledge lead to different levels of proof for the same theorem.	Proof is strictly dependent on the kind of reasoning you made	1
	Because many times it is possible to take different ways to prove something. All depends on the knowledge a person has and also on the ways he/she has been taught to reason.	1
	There may be more than one correct proof; some may be very artificial for it is difficult to find them	1
	I think there are theorems which have more than one correct proof, because these proofs can be built using different mathematical tools, sometimes more sophisticated, sometimes less, but also because they are situated in different mathematical contexts. (You may find a theorem both in analysis and in geometry for example). This is the reason why the same theorem may have a two lines proof and another may have a two pages proof.	1
	It depends on your knowledge background, a competent person may proof a theorem in a complicate way, for example with more advanced knowledge, but sometimes you may prove a theorem with easier tools, and then, in my opinion, a person's creativity has a big influence on the way you make a proof	1

	I think there may be more than one proof for a theorem, the difference is in the fact that some may result easier respect with some others	2
	Because many times it is possible to take different ways to prove something. All depends on the knowledge a person has and also on the ways he/she has been taught to reason.	1
	There may be more than one correct proof; some may be very artificial for it is difficult to find them	1
	I think there are theorems which have more than one correct proof, because these proofs can be built using different mathematical tools, sometimes more sophisticated, sometimes less, but also because they are situated in different mathematical contexts. (You may find a theorem both in analysis and in geometry for example). This is the reason why the same theorem may have a two lines proof and another may have a two pages proof.	1

	It depends on your knowledge background, a competent person may proof a theorem in a complicate way, for example with more advanced knowledge, but sometimes you may prove a theorem with easier tools, and then, in my opinion, a person's creativity has a big influence on the way you make a proof	1
	I think there may be more than one proof for a theorem, the difference is in the fact that some may result easier respect with some others	2
	You may use several methods to make a proof; you may start from different points of views and reach the same thing. This depends on the knowledge and on the tools you have, and furthermore it depends also on what view point you want to prove (e.g., mathematical, physics)	1
	Because it is possible to reach the same conclusion making different reasoning	2
	Different people may find different ways to reach the proof of a hypothesis	5
	Because in the Sciences there are different kinds of reasoning, namely, different schools of thought. For example, the issue about the "zero" regarding its position in the real numbers or in the natural numbers	1

	In my opinion it is possible to reach a proof following different ways, sometimes there doesn't exist a correct proof but there may exist several correct proofs	1
	You may try different ways using your own knowledge	1
	Through the reasoning you may find different ways to reach the proof of a theorem.	1
	A proof may follow different paths depending on the kind of study and level of knowledge, but also on the inspiration of the person who is performing the proof. For example many mathematicians have tried to find different and unusual proofs for the Pythagoras's theorem.	1
	Because there always exist several procedures, and formulas to be applied to reach the statement. No doubts, we can distinguish between easier proofs and more tedious one.	1
	The theorem is a unique thing but the ways you may explain it are different.	2
	Sometimes you may prove a theorem both analytically and graphically	1
	There are several mathematical tools that allow to proof the same thing in several ways	1
	There may be several correct proofs for each theorem because: 1) you may use different tools (theorems, postulates, and so on) 2) each person proceeds in a proof as he or she thinks the better way is.	1

	In my opinion some theorems (I am not sure all of them) may be proved in different ways. For example I have tried (I hope successfully) to prove that among all the triangle with the same perimeter, the one with maximum area is the equilateral triangle, without using derivatives, limits (therefore, the study of a function), but with a "logic" reasoning on Erone's theorem	1
	The ways to prove something may be many, the fact is that usually you use the more intuitive and immediate one	1
	Some theorems may be proved with different methods	1
	Even through different "paths" logic-deductive, it is possible to prove the validity of a theorem, the important thing is those paths to be correct and real	1
Probably the student is aware there exist more than one correct proof of the same theorem; but at school he/she usually sees only one of them. The sentence "only one is taught" underlines the idea that generally students' experience with the approach to proofs is something passive, it is something that is taught as it is, just as final ready-made product, and not something that is built with students' collaboration.	I think there exist several ways to proof a theorem; but usually only one is taught	1

Several ways	I think to say "only one" is very restrictive. Euclidean Mathematics is based on five theorems…therefore a concept may be proved following several ways	1
It depends on the individual creativity and initiative.	Because mathematics is a very wide subject matter and depending on the person who is proving the theorem there exist different ways to reach the thesis starting from the same hypothesis, it depends also on the person's creativity and initiative.	1
Only two things are constant: the logic and the truthfulness of the statements. What remains depends on the individual's creativity and initiative, the important fact is that such characteristics be supported by a basic cultural background.	I think that the only constant of a proof is the logic and the truthfulness of the statements. Therefore, I think there doesn't exist a fixed scheme and that creativity and initiative are the basis of a brilliant intellect on condition that it is supported by a certain "cultural background" that allows to reach correct conclusions.	1
The only important thing is that has to be logically correct.	The only important thing is that the proof must be logically correct	1
More than one correct proof	There may be more than one correct proof	3
From different hypotheses for the same theorem, you will obtain different correct proofs	You may pose different hypotheses to reach the same thesis	1
		89

COMMENTS

Almost the totality of the students agrees with the fact that there may exist more than one correct proof for the same theorem.

Different are the justifications given by the students. Some of them seem to be influenced by their scholastic experience, in the sense that they legitimate the existence of more than one correct proof, because they saw it at school (a sort of authoritarian scheme).

Others start from the idea that existing different ways of reasoning and different tools (axioms, postulates, and so on), there must exist different ways to make a proof for the same statement

Furthermore, a proving process depends on our own knowledge, for this reason such a procedure may take different aspects, not only but also, different levels of knowledge lead to different levels of proof. Interesting is the fact that students seem to be aware of the existence of several correct proofs for the same theorem, but they meet just one of them during their scholastic career.

The sentence “only one is taught” underlines the passive character of the students’ learning process; usually proofs are presented to students as a ready-made product, instead to be involved actively in the construction of it.

Some of the most indicative excerpts are listed below:

- ✓ *Many times during high school I saw theorems proved in different ways but all correct*
- ✓ *Proof is strictly dependent on the kind of reasoning you made.*
- ✓ *Because many times it is possible to take different ways to prove something. All depends on the knowledge a person has and also on the ways he/she has been taught to reason.*
- ✓ *I think there are theorems which have more than one correct proof, because these proofs can be built using different mathematical tools, sometimes more sophisticated, sometimes less, but also because they are situated in different mathematical contexts. (You may find a theorem both in analysis and in geometry*

for example). This is the reason why the same theorem may have a two lines proof and another may have a two pages proof.

- ✓ *It depends on your knowledge background, a competent person may proof a theorem in a complicate way, for example with more advanced knowledge, but sometimes you may prove a theorem with easier tools […]*
- ✓ *You may use several methods to make a proof; you may start from different points of views and reach the same thing. This depends on the knowledge and on the tools you have, and furthermore it depends also on what view point you want to prove (e.g., mathematical, physics)*
- ✓ *In my opinion it is possible to reach a proof following different ways, sometimes there doesn't exist a correct proof but there may exist several correct proofs*
- ✓ *A proof may follow different paths depending on the kind of study and level of knowledge […]*
- ✓ *I think there exist several ways to proof a theorem; but usually only one is taught.*

5th question THE CONSTRUCTION OF A PROOF HAS TO FOLLOW A FIXED PATTERN. CREATIVITY CANNOT FIND ROOM IN THE CONSTRUCTION OF PROOFS.		
		1st year of college (out of 89)
TYPE OF JUSTIFICATION	STUDENTS' TRANSCRIPTS	N. OF STUDENTS
False. Why?		
The only constant factor is the logic and the truthfulness of the statement. All remaining depends on creativity and personal initiative that are a smart mind's characteristics. The important thing is that creativity be supported by a	I think the only constant of a proof is its logic and the truthfulness of the statements. Therefore I think there cannot exist a fixed scheme and that creativity and initiative are at the basis of a brilliant intellect a on condition that they are supported by a certain "cultural background" that allows to reach correct conclusions	1

creativity be supported by a cultural background. Therefore, creativity and personal initiative are acceptable only if based on legitimate knowledge. Creativity meant as freedom for the choice of the points of view to be adopted. Again, the limitations are not on the structure of the proof but on the concepts used for the proof; such concepts have to respect the rigor and the validity of mathematics. Creativity as a very difficult but at the same time amazing thing, if it is correct. Creativity as a tool to simplify steps otherwise complicated and boring. Again the common idea underlying all these answers is the respect for the rigor of mathematics. Finally, the common denominator in this first group of answers is that creativity and personal initiative are important, are accepted, and useful, but at the base nothing would be acceptable if there weren't the rigor that characterizes mathematics	Because, if you mean creativity in the sense of freedom to start from where you want, I think it is possible to do it, what it is important is to be able to prove what you want. Probably, the limitations are not much in the structure of the proof but in the concepts you may use. A rigorous proof uses abstract concepts because stillness, invariability in time of the proof must be guaranteed	1
	Creativity in mathematics is the most difficult thing, but also the most beautiful (if correct). It may simplify steps that are only mechanics therefore boring. What is fundamental, anyway, is the fact that mathematical rules have to be respected.	1
	A proof doesn't have to follow a fixed scheme, but the tools you use must have sense and must lead anyway to the right proof of a theorem	1
	Most of the proofs are based on past experience, but also fantasy and creativity may give a useful help, always following mathematical rules. Indeed, you may find very few identical proofs	1

There are more than one way to make a correct proof; therefore, among these different approaches creativity may be a possibility. It is possible to reach the same target in several ways; this depends on personal creativity and initiative. The two previous interpretations are different: the former states that there are several ways of approach to a proof, therefore one of these approaches is creativity and personal initiative. The latter says that creativity and personal initiative are the cause of the heterogeneity of the approaches. Each of us has different ways of thinking, therefore creativity and personal initiative must play their role.	A proof may follow different paths according with the kind of studies and the level of cultural background, but also following the personal inclination of whom that is making the proof.	1
	There may be more than one proof for each theorem, as I already said	1
	There isn't only one way to face a proof	1
	You can try several paths using your own knowledge	1
	Mathematics is a very wide subject matter, and depending on the person who wants to prove the theorem there exist several ways to reach the thesis starting from the same hypothesis, it also depends on person's creativity and initiative	1
	As I said for the previous question, a proof of a theorem may be not unique and anyway also inside of the same proof is possible to make variations, leaving room for creativity and personal initiative. We have to say, that it is true that not many students have this kind of skill	1
	There are many ways to prove a theorem. Therefore personal initiative and creativity are at the basis of a proof	5
	Sometimes, there exist many mathematical tools to prove something, so you may chose the one you prefer	1
	Because each of us may find more logic one step instead of another one	1
	Each of us look at the problem from a personal point of view and he/she may solve it in the way he/she believes to be more convenient	1
	The proof depends on whom who is making it	1

	Each of us has a different form of reasoning	1
Intuitions seem to be the ones that lead to progress of the science and the enhancement of the knowledge. The reasoning used for a proof must be linear, but linearity doesn't exclude creativity and personal initiative. Furthermore, intuitions and creativity are fundamental elements for the progress of the science Probably students think of what they have studied at school, great philosophers, or mathematicians etc.	It also depends on the personal inclination, with it I mean the intuitions too, and the ones of great thinkers have enhanced the overcoming of some limitations f Analysis.	1
	The construction of a proof must without doubt follow a linear reasoning, but this doesn't exclude creativity and personal initiative. It there were not personal initiative I think no science could make progresses.	1
	Even the history teaches us: "a spot of genius" may lead to a proof that is totally out of traditional schemes adopted to build a proof.	1
	Creativity is the characteristic that stimulates human beings to the continuous research, that leads to knowledge	1
Fantasy arrives before the reality	The fantasy arrives before the reality	1
Fundamental tools Intuition may be a tool useful for the construction of a theorem. Sometimes intuition and creativity may be the only tools to find a proof	Many times without intuition, creativity, and personal initiative you cannot find an efficient proof.	3
	You might need the intuition to build a theorem	1
	Sometimes you can find a proof only thanks to intuition and creativity, in this case the proof looks more amusing	1

Several proofs are fanciful we could almost say, absurd. Probably, students think of proof they have seen at school, such as Lagrange theorem, or Taylor theorem, the proof of the first derivative of the product, where some artifices are used. They often cannot see the relationship between the artifice and what is being proved; and they obviously ask: “Where is this come from?” Therefore students cultivate such an idea that to make proof you need fantasy, for this reason creativity is one of its components.	Many proofs are very fanciful, we can say absurd.	1
	For some proofs you need a lot of fantasy	1
Creativity, personal initiative, and intuition as fundamental tools for the construction of a proof. They represent tools necessary to reason, to inquiry, to look at the problems from different points of view. Furthermore,	I think that creativity and personal initiative are the most important tools in the construction of a proof, because they help to think of and to wonder about problems of different kind (even though later on some of them may result not useful) and creativity and personal initiative develop a capacity' of personal critical analysis	1

<table>
<tr><td rowspan="8">points of view. Furthermore, creativity and personal initiative enhance the sense of critique
Without creativity, we would remain stuck in prescribed rules; such rules may represent an obstacle to look beyond.

They are also considered tools employed to shed light on properties still unknown.

Creativity necessary to build new proofs, easier than the ones already existing.

Creativity as tool employed to think at 360 degrees; it is an instrument to explore new ideas, to find a more efficient proof.
Students think that books for example often are not the easiest ones, or the most efficient; therefore creativity may enhance such kind of things.

As tool to simplify more complicated processes.

Creativity is the base for the greatest discoveries

To follow fixed schemes is not enough.

The “eyes of the mind” may not be able to see a solution that is right in front of them.</td><td>Initiative and personality are part integrating in the construction of a logic process that would remain stuck in prescribed rules if it weren’t adapted to an “esprit de finesse”</td><td>1</td></tr>
<tr><td>Along with creativity and personal initiative I would add intuition, because without it, when you meet an obstacle, during the construction of a proof you wouldn’t know what to do anymore; instead, a creative and intuitive person might find a solution</td><td>1</td></tr>
<tr><td>Creativity and personal initiative may lead to the discovery of alternative proofs sometimes correct, sometimes not. Anyway, such proofs may be useful to shed light on some properties not yet found</td><td>1</td></tr>
<tr><td>Many times creativity and personal initiative are the ones that lead to the birth of a new proof that may be easier or more complicate than the previous one</td><td>1</td></tr>
<tr><td>It is exactly creativity that makes us to think at 360 degrees, and to explore several ways and methods for a proof</td><td>1</td></tr>
<tr><td>If it were like that the several sciences wouldn’t have been evolved. There may exist several ways to prove a thing therefore it is not useful to “fossilize” on only one procedure. Furthermore, to go through new paths might lead to the discovery of new theorems or anyway to a greater consideration on points previously little considered</td><td>1</td></tr>
<tr><td>Sometimes creativity helps to solve problems, also in Analysis</td><td>1</td></tr>
<tr><td>You need to find the most efficient scheme in any way but it is correct</td><td>1</td></tr>
</table>

Hypotheses vary from theorem to theorem; therefore it is not possible to apply the same method. Without creativity there wouldn't be progress. Creativity as another possible "reading key"	Because personal initiative may allow to reach the same conclusion from different start points. Not always the proof you find in the books is the easiest. The single with his/her creativity may build a path to follow that is easier	1
	Many times personal initiative and creations help to solve a proof in a correct way	1
	Often creativity and initiative may shorten proving processes very difficult	1
	Many times with smartness you may find faster methods that simplify proofs	1
	Mathematics is indeed one of the sciences where human genius is very important. Some very difficult proofs were born thanks of amazing intuition.	1
	It is thanks of famous mathematicians' creativity that many theorems have been discovered. Following fixed schemes cannot be enough, because sometimes you have the solution in front of your eyes but you cannot see it with the eyes of the mind	2
	You cannot follow a fixed scheme, due to the fact that the hypotheses are always different. If creativity and personal initiative couldn't find a place in the construction of a proof, there wouldn't be any progress (i.e.: new proofs of the validity of statements)	1
	Very often intuition has played in the history of the human development a fundamental role	1
	Because personal intuitions may be a reading key different from prefixed schemes	1
There is no mathematical science without creativity	There doesn't exist mathematical science without creativity and personal initiative	1

and personal initiative. On the contrary any machine could make any proof, that is not true, therefore intuition that is a human being characteristic is fundamental. Theorems and axioms may be fixed, but what leads to the construction of a proof is the intuition. A proof is not something fixed (in the sense unique) at all therefore, creativity and intuition are basilar tools that help the construction of a proof.	Otherwise any machine could do any proof. Intuition is fundamental	1
	They are two fundamental characteristics to prove a theorem	4
	Theorems and axioms must be “fixed”, but often it is intuition deriving from personal initiative that leads to the construction of a correct proof	1
	Because without creativity and personal intuition you could never reach a complete construction of a proof	1
	Very often intuition and some intelligent tricks are needed	1
	I believe that intuition and personal creativity may help in a decisive way the construction of a proof that is not absolutely a “fixed” thing	5
Rationality and logic non always have fixed schemes.	Not always do logic and rationality have a fixed scheme	1
Creativity as part of the process for the formulation of a hypothesis	Creativity is part of the formulation of an hypothesis	1
The problem to leave the rationality that represents something certain, to approach creativity and personal initiative that represent uncertainty.	I always heard my teachers saying that personal initiative and creativity are important, I agree but my extremely rational personality prefers always and anyway a fixed schema	1
For new things we need new ideas, new methods, something not seen before, even though what we already know has its importance.	If I have to prove a thing I have never proved before, it is more than logical to use methods never used before, not forgetting the ones previously used	1
Several star points for the construction of a proof, therefore creativity and personal initiative are necessary.	Many times to build a proof you may start from different points (I believe) and in my opinion even if there might be a fixed scheme, creativity and personal initiative are always important.	1

Proofs as product of creativity and personal initiative.	Many proofs are the product of creativity and personal initiative	4
No universal scheme because any problem is different from the other	Being any problem different from the others, it would be wrong to think to solve it adopting procedures that follow a universal scheme	1
Creativity and personal initiative as tools to communicate and to make oneself understand	Because it is necessary, in order to make others understand, to use any kind of known tool, therefore creativity and initiative are at the basis of that.	1
The idea you may start from an intuition to prove the "algebraic" validity of an idea.	You may start from a more intuitive idea to proof algebraically the validity.	1
		80
True. Why?		
Mathematics as an applied science, therefore creativity and personal initiative cannot be employed	Unfortunately mathematics, in my opinion, cannot be invented, but it is applied. Therefore, apart from intuition (that anyway takes me to apply a preestablished procedure), I think that fantasy, and creativity, are not involved in mathematics.	1
Mathematics as a universal science that must use universal tools, understandable by everybody. Creativity and personal initiative cannot be used because are subjective.	Because they are tools of mathematics and they must follow a standard procedure in order to be universal and of easy use	1

Creativity and personal initiative cannot give any contribution to a proof	Personally, I don't see how creativity and initiative may give a contribution to the proof of a theorem or of a statement	1
Creativity and personal initiative may be employed only if based on scientific base	You may give space to intuition and creativity, but always on scientific bases	1
There are fixed rules that cannot be changed.	Even there may be several ways to prove you cannot change the rules	1
	There are some rules that has to be followed	1
Creativity is part of the formulation of a hypothesis; therefore, it cannot take part of the proving process.	I believe that creativity is part of the formulation of an hypothesis	1
There are no inventions; a proof is based on concrete things.	You don't have to invent things, but they must be proved based on concrete principles	1
Mathematics is a whole of fixed rules and schemes that must be followed with rigor.	A proof is a mathematical procedure that doesn't leave space to conjectures or creativity in the sense that any employed procedure must follow laws that are in a certain way and that cannot be in any other way. All you use for a proof is regulated by mathematical laws	1
		9

<u>COMMENTS</u>

80 students out of 89 claims that creativity and personal initiative are fundamental parts of a proving process. Many different justifications have been given to explain such a choice. For example, creativity and personal initiative are fundamental but are acceptable only when they respect a sort of rigor, peculiar characteristic of mathematics science; furthermore, creativity can be taken on consideration when is based on a cultural background and on recognized knowledge. Always in this case they underline that the limitations about the rigor are not related to the proof's structure but to the concept to be used.

For other students creativity and personal initiative are considered as a smart mind's characteristic. Furthermore, the aforementioned skills have to be considered possible tools of a proving process, because there is no only one way to approach a proof, namely among the different ways to tackle a proof there is intuition and creativity.

Nevertheless, there are two different explanations about this issue; some students justify the use of creativity and intuition arguing that they are just one of the several methods which can be used; others state that there may exist several ways to approach a proof because of creativity and intuition.

Students consider that many times proofs are very difficult; creativity and intuition may help to approach such a process in a easier way, not only but they enhance scientific progress and new knowledge. A possible explanation might be that students recall their scholastic knowledge about great philosophers, mathematicians and tinkers of the history.

In other cases creativity and intuition may represent the only tools useful to build a proof; or an important tool that allows to look at the problem from different points of view; to reason at "360 degrees", to look beyond what the "eyes of mind" may see. Prefixed rules may become a cognitive obstacle that may be overcome by intuition or personal creativity that also enhance the development of sense of critique**.** Therefore, creativity and intuition as an instrument of exploration, of construction of new knowledge, it is considered as a "reading key".

Furthermore, creativity and intuition are necessary to enhance fantasy; for some students fantasy is an important component in the process of proving, because many proofs are very artificial, and in order to find such artifices you need a lot of fantasy. Probably students think of proofs like the one for Lagrange theorem, Taylor theorem, or the first derivative of the product and so on.

Finally, there are no fixed schemes; any problem is different to another one, for this reason we have to employ creativity and personal initiative. In addition no machine may build any kind of proof therefore creativity is needed. To conclude, creativity and personal initiative are the tools to communicate with the others, and to make one understand.

The remaining students, exactly 9, argue that creativity and personal initiative cannot be part of a proving process. First of all because mathematics is an applied science and the previous two cannot be applied; mathematics is a universal science that must use universal tools, understandable by everybody, and creativity cannot be considered universal, on the contrary it is subjective.

Furthermore, in mathematics there are fixed rules that cannot be changed. For some students creativity is part of the formulation process of a hypothesis, therefore it cannot be part of a proving process. It seems that the formulation of a hypothesis and the process of proving be two disconnected things.

Finally, mathematics is seen as a whole of fixed rules and schemes that must be followed with rigor.

Some student's excerpts follow:

- ✓ *Because, if you mean creativity in the sense of freedom to start from where you want, I think it is possible to do it, what it is important is to be able to prove what you want. Probably, the limitations are not much in the structure of the proof but in the concepts you may use. A rigorous proof uses abstract concepts because stillness***,** *invariability in time of the proof must be guaranteed*
- ✓ *Creativity in mathematics is the most difficult thing, but also the most beautiful (if correct). It may simplify steps that are only mechanics therefore boring. What is fundamental, anyway, is the fact that mathematical rules have to be respected.*
- ✓ *There are many ways to prove a theorem. Therefore personal initiative and creativity are at the basis of a proof*
- ✓ *Even the history teaches us: "a spot of genius" may lead to a proof that is totally out of traditional schemes adopted to build a proof*
- ✓ *Many times without intuition, creativity, and personal initiative you cannot find an efficient proof*
- ✓ *I think that creativity and personal initiative are the most important tools in the construction of a proof, because they help to think of and to wonder about problems of different kind (even though later on some of them may result not*

useful) and creativity and personal initiative develop a capacity' of personal critical analysis

- ✓ *Creativity and personal initiative may lead to the discovery of alternative proofs sometimes correct, sometimes not. Anyway, such proofs may be useful to shed light on some properties not yet found*
- ✓ *It is exactly creativity that makes us to think at 360 degrees, and to explore several ways and methods for a proof*
- ✓ *It is thanks of famous mathematicians' creativity that many theorems have been discovered. Following fixed schemes cannot be enough, because sometimes you have the solution in front of your eyes but you cannot see it with the eyes of the mind.*
- ✓ *Theorems and axioms must be "fixed", but often it is intuition deriving from personal initiative that leads to the construction of a correct proof*
- ✓ *Being any problem different from the others, it would be wrong to think to solve it adopting procedures that follow a universal scheme.*
- ✓ *There are some rules that has to be followed*
- ✓ *A proof is a mathematical procedure that doesn't leave space to conjectures or creativity in the sense that any employed procedure must follow laws that are in a certain way and that cannot be in any other way. All you use for a proof is regulated by mathematical laws*

6th question A PROOF IN CALCULUS HAS THE FOLLOWING ROLE	
	1st year of college (out of 89)
1 To convince somebody about the validity of a statement	1
2 To explain why a statement is valid	19
3 To establish the validity of a statement	22

1 e 2 To convince somebody about the validity of a statement **and** To explain why a statement is valid		7
1 e 3 To convince somebody about the validity of a statement **and** To establish the validity of a statement		2
2 e 3 To explain why a statement is valid **and** To establish the validity of a statement		26
1 e 2 e 3 To convince somebody about the validity of a statement **and** To explain why a statement is valid **and** To establish the validity of a statement		8
Something else (Specify)	**1 +** To make hope that you didn't waste your time for something without sense	1
	1+2+3+ All these things, but only in a certain sense. For example, in the history of physics before the "revolution" brought by the relativity, all proofs were valid exactly "in function" of the conceptions (in this case "space" and "time") of time, therefore you can say that they establish the validity of a statement until a change given by a new revolution	1
	2+3+ something else Jacopo explains for each points he has chosen his point of view. He has chosen 2. To explain why a statement is valid, and he writes: *It is interesting the relationship between the several hypotheses and the related steps of the proof.* 3. To establish the validity of a statement, and he writes: *It is important but only for whom who is dealing with mathematics at high levels. The statements given to us as students we know in advanced that are true* Something else, and he writes: *It is necessary to underline in which cases a theorem may be used, with particular attention to the control of all hypotheses. Very often you make the mistake to use a rule without verifying the hypotheses (I say it for personal experience)*	1

To convince somebody and **themselves** about the validity of a statement	1
	89

The predominant idea about the role of a proof is the following: it must explain and validate.

Appendix B: Transcripts of students' protocols

Alice and Roberta (fixed point problem)

R1: **RA**: domain from [0,1] to [o,1]

R2: **R:** *fixed point on the bisector line and therefore…*

R3: **A:** (she draws the bisector line) *therefore this is the (1,1) and (0,0).*

R4: **R** *The fixed point must be between these two points …*(and she signs the two points (0,0) and (1,1) going along the bisector line)

R5: **A**: *Exactly…but it could have only these two points [(0,0), (1,1)]; if it were in this way* (and she signs a concave function over the bisector line) *therefore there is a fixed point for sure, because there are these two points of the bisector line* (and she signs (0,0) and (1,1))

R6: **R:** *eh…no, because the function could start from here and from here*

R7: **A:** *you are right, it is true; it is defined from 0 to 1…*

R8: **R:** *oh yes...the function starts from 0 and then there is a point here for sure (she underlines the segment from 0 to 1 on x-axis) and it arrives at x=1, therefore there is also a point here for sure (she underlines the side of the square of vertexes (1,0) and (1,1))*

R9: **A:** *oh right...then it has to intersect the bisector line for sure…suppose that it does like that…*

R10: **R:** *hmmm…the function must have a fixed point for sure…because it has to pass from here to there*

R11: **A:** yes…it must go through for sure…then the point of abscissa x=0 could have y=0 then it would have a fixed point or it could be >0 then it doesn't have…and the point x=1 could have y=1 and then it would have a fixed point or ≠1 then it would not have the fixed point.

R12: **R:** then it would intersect in the middle…I mean in a point whatever (and both Alice and Roberta draw hypothetical functions)…I was thinking…only one…it could have more than one…

R13: **A:** yes probably yes…(Alice draws a kind of sinusoid)

R14: **R**: but in a function for each x must correspond a y…

R15: **A:** yes…this is always a function

R16: **R:** ah…yes yes…exactly

R17: **A:** therefore at least there is one point for sure

R19: **R:** yes there must be for sure…because anyway one point here and one point here (she signs the extremes) …here I get confused because we have more than one…

R20: **A:** you have more y…

R21: **R:** yes exactly

R22: **A:** I mean, given a y there are several x corresponding to it, but not that for an x several y correspond

R23: **R:** yes exactly…

R24: **A:** we would better write something

R25: **A/R:** eh yes…there is one for sure

They start writing…

R26: **R:** then this function must have a point here and one there (they sign the two sides of the square) for sure

R27: **A/R:** (they start organizing a proof going through the fundamental steps they touched in the construction of their conjectures)

R28: **A/R:** *then the function must start from 0 and have f(x) on this side and arrive at the point of abscissa x=1 and f(x) on this side then…there is the bisector line that goes through (0,0) and (1,1)*

R29: **R/A:** therefore we write…

R30: **A:** I was thinking…there must exist a point of abscissa 0

R31: **R:** exactly…and the y…

R32: **A:** and the y…

R33: **R:** the y between 0 and 1

R34: **A:** (Alice writes) *then P(0, 0≤y ≤1) because the domain…*

R35: **R:** *it is defined from 0 to 1*

R36: **A:** dom=[0,1] and cod=[0,1]

<u>R37</u>**: R:**and it must be the same for the point of abscissa 1

<u>R38</u>**: A:** (Alice writes) *it must exist too…P_1 (1, 0≤ y ≤1), I would start with the limit cases, P(0,0) and P_1 (1,1) or when* (she goes with her finger from the point (0,0) along the segment 0-1 on the y-axes, and she does the same with the punt (1,1) downwards)

<u>R39</u>**: R:** I understood what you mean

<u>R40</u>**: A:** the cases where P and P1 are the fixed points (and she writes P(0,0) e P1(1,1))

<u>R41</u>**: R:** (she starts saying…signing possible functions on the graph) if it did like that (and she signs a concave increasing function) then there are two, if it did like that…there would be only one and in the other way there would be more than one, there here is one for sure.

<u>R42</u>**: A:** Therefore this (P=(0,0) and P1(1,1)) is not the limit case because we have two fixed points

<u>R43</u>**: R:** let's explain why…

<u>R44</u>**: A:** there could be fixed points every time that the function intersects the bisector line…but then there could be infinite fixed points.

<u>R45</u>**: R:** well we can't know this, but we know for sure that there is one fixed point (at least)

<u>R46</u>**: A:** once we have proved that there is one we are done, we don't have to prove that there is more than one fixed point.

<u>R47</u>**: R:** now let's do the cases where the function does not go through (0,0) and (1,1) but a point over here (and she signs the segment 0-1 on the y-axis)

At this point they write on their protocol:

If the function f(x) goes through P(0,0), a fixed point is P; There could exist other fixed points in the case that the function intersects the bisector line.

In the same way, if the function goes through the point P(1,1). In all other cases the function will have to go through a point with abscissa 0 and a point of abscissa 1 (for hypothesis). In these cases the ordinate of the point with abscissa 0 will have to be 0 ≤ y ≤ 1, and the ordinate of the point with abscissa 1 will have to be 0 ≤ y ≤ 1. Being the function continuous for any path satisfying the aforementioned conditions will have to *intersect the bisector line in at least one point (on the bisector line lie all the fixed points).*

R48**: A:** anyway…we could also prove it taking a square…a point running on a side and another point running on the opposite side…then we link the two points…you can do a non-linear path too, and you see that the function always intersects the bisector line

Serena and Francesca (limit problem)

R1: **S**: : h goes to zero…x_0+h…

They immediately draw the graph visualizing x_0, $x_0 + h$, $f(x_0)$, $f(x_0 + h)$…

R2: **S**: *$f(x_0+h)$*

She looks at it on the graph

R3: **S**: *when $h \to 0$ this gets closer here and also $f(x_+h)$*

R4: **F**: this difference is exactly…

R5: **S**: it goes close to $f(x_0)$…

R6: **F**: exactly…

R7: **S**: anyway, this difference goes to zero…and if we separate them? $\frac{f(x_0+h)}{2h}$…

$f(x_0+h) \to f(x_0)$…

R8: **F**: $\frac{f(x_0+h)}{h} - \frac{f(x_0)}{h}$ *and then we add it…*

R9: **S/F**: *let us write it down better:* $\lim_{h\to 0} \frac{1}{2}\left(\frac{f(x_0+h)}{h} - \frac{f(x_0)}{h}\right) + \frac{1}{2}\left(\frac{f(x_0)}{h} - \frac{f(x_0-h)}{h}\right)$

R10: **F**: *this* (referring to the first parenthesis) *is our $f'(x_0)$ therefore* $\frac{1}{2}f'(x_0)$

R11: **S**: *that thing there* (referring to the second parenthesis)…

R12: **F**: *it will be a difference quotient as well…because if you look at the drawing…from this you take off this and divide by h; from that you take off this and you subtract h, therefore the difference should be the same thing…*

R13: **S**: *then…1/2 $f'(x_0)$ – 1/2* $\lim_{h\to 0}\left(\frac{f(x_0)}{h} - \frac{f(x_0-h)}{h}\right)$ *this goes to zero…*

<u>R14</u>: **F**: hmmm…

<u>R15</u>: **S**: in my opinion is wrong…ah…but wait…here there is –h therefore this becomes +…then f'(x_0)…

[…]

<u>R19</u>: **F**: *yes…also because basing on my intuit I would have said that the limit would go to f'(x_0)….therefore* $\left(\frac{f(x_0)}{h}-\frac{f(x_0-h)}{h}\right)$ *is the difference quotient*

Francesca repeats it to me

<u>R20</u>**:** **F**: We did it very algebraically…and we said…first we add $\left(\frac{f(x_0)}{h}\right)$ *and then we subtract it…first we take out 1/2 …*$\frac{1}{2}\left(\frac{f(x_0+h)}{h}-\frac{f(x_0-h)}{h}\right)$ *I add and subtract* $\frac{f(x_0)}{h}$ *therefore here taking it out, I have exactly the difference quotient, thus I have f' (x_0) here…*

<u>R22</u>: **I**: *here can I say that it is f'(x_0)?*

<u>R23</u>: **F**: $\left(\frac{f(x_0-h)-f(x_0)}{h}\right)$ *let us change the signs…*$-\frac{1}{2}\left(\frac{f(x_0)-f(x_0-h)}{-h}\right)$ and we said…

<u>R24</u>: **S**: *that the difference quotient can be* $\left(\frac{f(x_0+h)-f(x_0)}{h}\right)$ *but also* $\left(\frac{f(x_0-h)-f(x_0)}{-h}\right)$

<u>R25</u>: **I**: *Why?*

<u>R26</u>: **S**: *because h goes to zero therefore –h goes to zero and thus even this is f'(x_0), then* $\frac{1}{2}f'(x_0)+\frac{1}{2}f'(x_0)=f'(x_0)$

Matteo and Marco (fixed point problem)

R1: **Matteo**: this function is in the middle, I would say…I mean…it goes from here to there

R2**: Marco**: from 0-1 to 0-1

R3**: Matteo**: if the function starts from 0 and goes up and goes down, it takes all the values one time…and we have two fixed points.

R4**: Marco**: The fixed points are these, then?

R5**: Matteo**: the function must have fixed points, if we find such a function that doesn't have fixed points, we have solved the problem; on the contrary, if we have to prove that it has a fixed point, then it is amore difficult.

R6**: Matteo**: I suppose that if the problem asks, the function will have a fixed point.

R7**: Marco**: How can we find this fixed point?

R8**: Marco**: a fixed point is here, another one is here…

R9**: Marco**: therefore, the fixed points are those that have y = x?

R10**: Matteo:** I would say yes…I would say that the fixed points are on…y = x…and if our function must assume all the value of the image in such a way if it is continuous it must go through this line…there will be a point for sure…

R11**: Marco:** we know that starts from x=0 and arrives at x=1, it has to arrive here.

R12**: Matteo:** supposing that it does not have to intersect this thing, and given the fact that it must take all the values from 0 to 1, the value with x=0 must exist, if for this x=0 y were equal to 0 we would have a fixed point, therefore it does not work, then y must be different to 0 and at this point we would have one of these points here. When we want to go to x=1 or y=1 and we don't want to, therefore y ≠ 1, then we have one of these points here and one of these points here to go from here to there in any way we have to go through here and therefore any function which brings one of these points here to a point there must intersect the bisector line, for sure…

R13**: Matteo:** in my opinion we should think of a counterexample, somebody saying that it is possible to pass, I have to find the way to prove that we can't pass without intersecting the line, at

R14**: Matteo:** io suppongo che in qualche modo la $x = 0$, $y = a$ con $a \neq 0$, poi abbiamo $x = 1$, $y = b$ con $b \neq 1$, prendo una f(x) qualsiasi..ok…abbiamo 3 casi: $a > b$ questo è il punto a e questo è il punto b e c'era un teorema, forse Lagrange o qualcosa del genere che ci assicurava che intersecava qualcosa…che c'era qualcosa che intersecava qualcosa..se sono uguali o se uno è più alto dell'altro

R15**: I:** Matteo tries to explain to Marco

R16: **M:** we have to prove that f (x) intersected with $y = x$ is not empty, different to the empty set. We have to prove that it is possible to go from here to there without intersecting the bisector line, but if $a > b$ taking a as the point where $x = 0$ and that lies on the upper side of the bisector line, b the point where $y = 1$ and b lies on the lower side of the bisector line there must be a point between the two where the $x = y$…there must be for sure and I can do the same thing changing the position of the two points respectively…or collocating them at the same height…I have to write it down in formal way…

I: Now they explain the proof to me

R17: **Matteo**: by contradiction we take 'a' that is greater and $\neq 0$ and 'b' minor, now we say by absurd it doesn't go to, at this point 'a' will take in this point here any point in the middle and that $a \neq y$, therefore a point in which $y > x$ always because in a first moment we said that it was greater therefore y must be greater than x and in this other little point here and here and here it will always be greater strictly greater we arrive here where it must be greater than x, at this point we have to take all these points here; its value in 1 cannot be less than 1, equal 1 or more than 1 because it must stay in this interval here, therefore it is absurd.

Alice and Maggialetti (limit problem)

R1: I: *they read the text*

R2: **A:** *at the end…it is the difference quotient…only that there is 2h instead of h…*

R3: M: eh yes…

R4: **A:** *no…wait…*

R5: M: but…this part here the difference quotient is not like that (he refers to $f(x_0-h)$)

R6: A: no in fact

R7: **M**: *it is similar to the difference quotient…then…the difference quotient is…*(they think for a while and then they conclude) $\frac{f(x_0+h)-f(x_0)}{h}$ *…yes…yes it is similar to…but there is not* $f(x_0-h)$

R8: I: they ask me if it is true that the difference quotient is

$$\frac{f(x_0+h)-f(x_0)}{h}$$

At this point they write $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$

R9: **A:** *I write also the difference quotient.*

R10: M: with the definition of limit…like we write this…and we take $\forall\varepsilon$

R11: A: b ut we have two

R12: **M:** o*h yes...in other words we have the limit of two functions, I mean, the limit of* $\frac{f(x_0+h)}{2h}$ *minus the limit of* $\frac{f(x_0-h)}{2h}$ *and we cannot say that is the limit of the difference, so to speak, we take the result of this…*

R13: **A:** *but with the limit...what we arrive to say? Because…at the end…we know how to calculate this limit…we know that the function is defined and differentiable, therefore we know that is continuous, then we don't need to do all the calculation of the limit…*

R14: M: you are right…that's true

R15: A: h that goes to zero…

R16: M: differentiable…therefore continuous

<u>R17</u>: **A**: *You know what we can do? In* $\frac{f(x_0+h)-f(x_0)}{h}$ *there was the graph to show that it was the slope…*

<u>R18</u>: **M***: yes…of the line…*

<u>R19</u>: **A**: *perhaps this is related to the slope but shifted up or down…*

<u>R20</u>: **A**: *I mean…when h* $\rightarrow 0$*…do you remember the graph?*

<u>R22</u>: M: no…but if we take this point here it will be x_0 and this $f(x_0)$

<u>R23</u>: A: this distance is h therefore this is x_0+h

<u>R24</u>: M: therefore this is f(x_0+h)…ah…and this is f(x_0+h) – f(x_0) (and they sign on the y-axis such difference)

<u>R25</u>: **A**: *then…when h* $\rightarrow 0$*…oh yes…this becomes the tangent line in this point here*

<u>R26</u>: M: yes right

<u>R27</u>: A: I mean…what does the chord do?

<u>R28</u>: M: namely, this is the slope of the tangent line to the function…

<u>R29</u>: A: exactly…was the drawing in this way?

<u>R30</u>: M: I don't remember…anyway we have taken a function (he seems to be sure of what they did)

<u>R31</u>: **A**: *now let us try to draw this (* $\frac{f(x_0+h)-f(x_0-h)}{2h}$ *)*

<u>R32</u>: I: at this point they build the function

<u>R33</u>: **A**: in my opinion this could work as a difference quotient…

<u>R34</u>: **M**: *but the difference quotient is the slope of the tangent line…*

<u>R35</u>: **A**: *yes…*

<u>R36</u>: **M**: *and there, it goes…here what does this (* $\frac{f(x_0+h)-f(x_0-h)}{2h}$ *) represent?*

[…]

<u>R41</u>: **A**: *It could represent the slope of the tangent line…*

<u>R42</u>: **M**: *the tangent line in which point…?*

<u>R43</u>: **A**: *We need to see in which point…then, if h goes to zero…let us see what happens when h goes to zero…it means that…here there is a distance of 2h…between* $x_0 + h$ *and* $x_0 - h$

<u>R44</u>: M: si

<u>R45</u>: **A**: *when h goes to zero, this becomes zero and goes to x_0, this one becomes zero and goes to x_0…therefore all the values go to x_0…while here (she refers to* $\frac{f(x_0+h)-f(x_0)}{h}$*)…too…at the end they always go to x_0…because the numerator when h goes to zero goes to…wait…goes to zero…*

<u>R46</u>: **M:** *here* (referring to the expression $\frac{f(x_0+h)-f(x_0)}{h}$) *it goes to…zero…oh…OK*

<u>R47</u>: Alice signs on the y-axis $f(x_0+h)-f(x_0)$ and $f(x_0)-f(x_0-h)$

<u>R48</u>: **A**: *then…here we have* $\frac{f(x_0+h)-f(x_0-h)}{2h}$ *... f(x_0-h) is equal to f(x_0+h)-…*

<u>R49</u>: **M**: *minus* $\frac{f(x_0)}{h}$*...*

<u>R50</u>: I: they think if they can make a graphically sense of f(x_0-h)…but they realize they don't arrive at anything, since they arrive at an identity, therefore they change strategy

<u>R51</u>: **A**: *but we can write it as...I mean the limit of this one...* $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$ *...as a matter of fact we know the numerator, we can write it as addition and subtraction of limits in such a way to have inside of the expression* $\frac{f(x_0+h)-f(x_0)}{h}$

<u>R52</u>: **M**: *OK...you take out* $\frac{1}{2}$*…*

<u>R53</u>: I: Alice writes $\frac{1}{2}\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{h}$

<u>R54</u>: **M**: *do you want to have the difference quotient?*

<u>R55</u>: I: at this point they think for long time

<u>R56</u>: **A**: *we could write…(she adds and subtracts f(x_0)) and then we separate it…* $\frac{1}{2}(\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}-\lim_{h\to 0}\frac{f(x_0-h)-f(x_0)}{h})$ *the first become f'(x_0) and the second one?…I don't know…*

R57: **M**: *isn't it the difference quotient with the difference that there is a minus? Therefore it is the same thing but considered at the other side…*

R58: A: therefore it becomes ½ ($\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ + the limit …no…let's put - $\lim_{h\to 0}\frac{f(x_0-h)-f(x_0)}{h}$ …therefore this (referring to the first limit) is the first derivative

R59: M: and this one?

R60: A: it seems like another piece of the function

R61: M: I mean they are two…this represents this piece, and this represents this other piece

R62: A: yes, but then with the limit you go back here…

R63: M: that's true …

R64: **A**: *then it could be zero…I mean…in both cases you arrive at the slope of the tangent line here. Therefore, it is the same thing of doing the slope of the tangent line here, minus the slope of the tangent line always here…*

R65: M: you know…

R66: A: therefore doesn't it become zero?

R68: **M**: *yes. Zero.*

Daniele and Betta (limit problem)

Daniele draws a function and signs x_0, x_0+h, x_0-h, f(x_0), f(x_0+h), f(x_0-h).

R1: **D**: x_0+h...

R2: **B**: f (x_0)…

R3: **D**: in my opinion it is the same thing… when you do the limit of the difference quotient, you do $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ …this minus this over h…

R4:**D**: he signs on the drawing done on the protocol, this | divided by this —)

R5: **B**: because f(x0 + h)...

R6: **D**: minus f(x0)...is this

R7: **B**: Ah…OK…ours would be this over 2h…it is the same thing…

R8: **D**: therefore…it would be h→ 0…how much is this?…eh…it will be the slope of the tangent line…

R9: **B**: namely…the first derivative

R10: **D**: in x_0...

R11: **I**: if you should justify it rigorously?

R12: **B**: this is the same...

R13: **D**: because this limit is equal to limit for h going to zero of this…

R14: **I**: I didn't understand...

R15: **D**: because the limit of the difference quotient is equal to the limit of this (and he signs $\frac{f(x_0+h)-f(x_0-h)}{2h}$)

R16: **B**: this is equal to this (they indicate the two limits…)…we done it graphically

R17: **D**: I mean, we do this…it would be the ratio between this difference | and this one — and in our case it would be the ratio between this difference | and this one — , therefore, x0 + h –(x0 – h) that would be 2h…and this one that would be f(x0 + h) – f(x0 – h)...therefore, the limit for h that goes to zero would be…I mean both go to x0

R18: **I**: do you think this justification to be rigorous?

R19: **D**: Probably we didn't prove it..but in theory…I mean…

R20: **I**: Do you think the proof you have done at graphical level to be rigorous? In the sense…if you asked you…in a written proof you are asked to prove it in a rigorous way…you would stop here?

R21: **D**: at an intuitive level, yes…but in my opinion it is not a rigorous justification

R22: **I**: why?

R23: **D**: because if somebody explained it to me in this way…I wouldn't…

R24: **I**: you wouldn't believe him?

R25: **D**: no…I mean…but it seems to me to know it only in this way…

R26: **I**: (note: Daniele thinks)

R27: **D**: eh yes…anyway it is correct…I mean, the difference quotient would be this chord …namely, it would be the tangent line of this angle, right? The difference quotient…therefore, for h that goes to zero, this…this chord…shrinks more and more till when it becomes a point and it is the tangent line in that point…in this case it is the same thing

R28: **I**: If you were told in this way…it would be enough for you? Would you be convinced if one of your classmates explained it to you in this way? Would you say….ah OK…yes, yes…or would you have some doubts?

R29: **D**: we should write it down…

R30: **I**: how do you write such a thing?

R31: **D**: firstly, if I have an equation and I do the limits of the both parts…it is the same thing…

R32: **B**: therefore, if you prove that this is equal to this (namely, $\frac{f(x_0+h)-f(x_0-h)}{2h}$ and $\frac{f(x_0+h)-f(x_0)}{h}$)

R33: **D**: eh…therefore…yes but I must…it would be…

$$2\frac{f(x_0+h)-f(x_0)}{h}=\frac{f(x_0+h)-f(x_0-h)}{2h}2$$

And they simplify in the following way

$$2\ \frac{f(x_0+h)-f(x_0)}{\cancel{h}}=\frac{f(x_0+h)-f(x_0-h)}{\cancel{2h}}\cancel{2}$$

R34: **I**: but then you have already given for sure that this and this one are equal…

R35: **D**: ehm…yes…

R36: I: no, you have to prove it. I thought you would want to prove that

$$\frac{f(x_0+h)-f(x_0-h)}{2h}=\frac{f(x_0+h)-f(x_0)}{h}$$

R37: **D:** I wanted to prove that when this becomes zero even this becomes zero…(note: he makes an expression like to underline he knows to have said something just to say something)

R38: **I**: Ah…I thought you wanted that $\frac{f(x_0+h)-f(x_0-h)}{2h}=\frac{f(x_0+h)-f(x_0)}{h}$

R39: **D**: : yes…but you are right! I already thought to be true the equality…then, I looking for…no, no

R40: **B**: but if you write the two expressions this from one side of the equal sign and this from the other side and for each thing you show what correspond on the graph…ah yes…but in this way we start always from the figure…

R41: **D**: I can't write…it is not correct…but…when this becomes zero…if it is equal to the other…even the other one has to become zero…

R42: **I**: senx for x going to zero becomes zero, x for x going to zero becomes zero, but they are not equal…

R43: **D**: ah…that's true…therefore (they continue to manipulate the expression and arrive at the second row $2f(x_0+h)-2f(x_0)=$….after that looking at the graph they arrive to say that $f(x_0+h)-f(x_0-h)=2f(x_0)$ therefore I substitute here and I look what happens, it would be $f(x_0 + h) – f(x_0 – h) = 2 f(x_0)$…therefore this plus this must be equal to this…now I substitute…

R45: **D**: can't I put this equal to c (perhaps I said already said it) and this is equal to c?

R46: **I**: the limit?

R47: **B**: not only what we have inside…namely…

R48: **D**: eh…no no…this is a quotient…I mean it would be this and this…they can be also different…but then the limit is the same…

R49: **B**: yes…but we have proved that this is equal to this…namely, according to our drawing…ok that they could be different…but…let's try to do as she says (note: I have told them that to prove the equality between two things you have to manipulate both separately till when one becomes equal to the other).

R50: **D**: I have understood…but how do you do?

R51: **B**: I mean…let's multiply, divide by 2…something to make it equal, do you know what I mean?

R52: **D**: then…wait…it would be…this minus this divided by 2 (note: he is looking at the graph signing |) $\frac{f(x_0+h)-f(x_0-h)}{2}$ + $f(x_0 – h)$...we need it…because this (note: $\frac{f(x_0+h)-f(x_0-h)}{2}$) it would be this plus this which would be $f(x_0)$...and therefore we substitute here… $f(x_0 + h)$ minus this…it would be $2(f(x_0 + h) - f(x_0 - h) + f(x_0-h)$ divided by this and multiplied by 1 over h

R54: **D**: graphically it would be $\frac{f(x_0+h)+f(x_0-h)}{h}$ it would be this distance, right? Then…we know…this would be $\frac{f(x_0+h)+f(x_0-h)}{2}$ and to that we added $f(x_0+h)$...and making the calculations it becomes this…

R55: **I**: do you think of this proof of a rigorous proof? Therefore, with this you would be sure that such limit corresponds to the first derivative.

R56: **D**: the calculations are correct…but if the function were like that…no…this would be x_0…this x_0+h then I have $f(x_0)$ and $f(x_0+h)$…the first rate gives me this chord right? It gives me the tangent of this angle…the slope of the line through two points…and in the other it gives me this…right?

R57: **I**: yes...

R58: **I**: Daniele has some perplexity about the drawing…something doesn't sound correct…then I make a third drawing…

R59: **I**: Daniele is surprised by the fact that the equality they make before the passage to the limit would bring to parallelism between two lines that go through a same point

R60: **D**: it is obvious that passing to the limit these two points coincide…namely I can't write that $\frac{f(x_0+h)+f(x_0-h)}{2h} = \frac{f(x_0+h)+f(x_0)}{h}$ because it is false…

R61: **D**: in fact if $f(x) = g(x)$ then the limit of $f(x)$ is equal to the limit of $g(x)$ but not the contrary…

R62: **B**: instead we…proving it in this way we proved the equality

R63: **I**: because what did you say?

R64: **B**: $f(x_0) = \frac{f(x_0+h)+f(x_0-h)}{2} + f(x_0-h)$…we have seen it graphically…but at the end in this way…for sure…

R65: **I**: $f(x_0)$ you said…but why did you assume that…

R66: **B**: these two are equal…

R67: **D**: ah, that's true…in fact…

R68: **I**: It's not said that f(x_0-h) and f(x_0+h) would be equidistant from f(x_0)

R69: **D**: we did a drawing that misled us

R70: **I**: the drawing misled you but it also helped you to understand the mistake

R71: **D**: I mean it is valid only if it is linear

R72: **B**: therefore it doesn't work…therefore f(x) is not equal…and therefore algebraically we can't do it…

R73: **I**: yes…but not like this…

R74: **D**: in our case we have to write that f(x_0) was equal to this plus this that would be…I would like to write f(x_0) in function of these two…

R75: **B**: but we don't have to prove that this is equal to this…

R76: **I**: exactly…you continue to stuck with the idea to prove that this is equal to this…you said an important thing about which implication is true and which is not…therefore these two have the same limit but probably they are not equal

R77: **D**: but now neither the graphic one convinces me anymore…because we used the symmetry respect to f(x0)…no, no…that one is true

R78: **B**: ah…yes yes…

R79: **I**: what has been the conjecture rose by the graph? Therefore…from the graph you said…probably is f'(x0)

R80: **D**: yes...

R81: **I**: start from that conjecture, namely limit of $\frac{f(x_0+h)+f(x_0-h)}{2h}$ = f'(x_0)

R82: **D**: we have to say that here…I mean…but it is always the middle point of this segment

R83: **B**: if we wanted to find f(x_0) in function of something…but related to this figure…

R84: **D**: now I am going to say something stupid…but at the numerator we have a function that goes to zero for h going to zero, right? And also below…therefore we have to prove that this has the same order of this…then we have a c…

R85: **I**: but who told you that it is the first derivative?

R86: **I**: Daniele e Betta start thinking how to manipulate algebraically the starting expression…Daniele starts writing something and asks me if it is correct…they added and subtracted $f(x_0)$ to $\frac{f(x_0+h)+f(x_0-h)}{2h}$ then they conclude that the first addend is $f'(x_0)$ differing only for the factor ½, and the second I ask why it is also $f'(x_0)$

R87: **D**: we justified it graphically…

Daniele and Francesca (fixed point problem)

They immediately draw the bisector line as the line of the fixed points

R1: **Fr.:** there must be an intersection between the function and the bisector line

R2**: I:** Daniele rereads the text.

R3**: Fr.:** if there is the fixed point there must be the intersection with the bisector line, for sure

R4**: Dan:** there are two for sure…ah no…there is one for sure

R5**: Fr:** if there weren't (fixed points) it (the function) would stay all over or all under the bisector line…the only case would be if the bisector line were the asymptote of the function…

R6**: Dan:** but it is not possible

R7**: Fr:** …but it is not possible because it is continuous…

R8**: Dan:** : it is not possible because 1 is between…I mean…the function in 1 exists…that is, here it is included…(ndr: he writes a square parenthesis on 0 and on 1 on the x-axis and he does the same thing on the y-axis)

R9**: Fr:** therefore the bisector line cannot be an asymptote, and then if it is not an asymptote it must cross it for sure…

R10: Fr: (talking to I) *probably we answered…if A is a fixed point it must have an intersection with the bisector line…the only case for the contrary is if the bisector line were the asymptote of the function…but, if the function is defined from [0,1] to [0,1] included…the function is defined in 1 too, therefore at the most the point is (1,1) or it crosses it.*

R11**: I:** Daniele draws the function

R12**: I:** this, though, doesn't work because it doesn't take all the values form 0 to 1.

R13**: Dan:** ah it must take all the values

R14**: Fr:** ah but then it has two for sure…even more

R15**: I:** why two for sure?

R16**: Dan:** (he draws several functions, then he realizes that it is not like that) *anyway, there is one fixed point for sure…if it must take all the values and if we make it start from here…if it must take all the values it must start from this point…from this…this…because it can't come back…to take all the values it must start from the maximum up to the minimum…if we think of that theorem where if you have a point here and one here it must go through here, for sure*

R17**: Fr:** it is the Theorem of the Zeros…

R18**: Dan:** (he is repeating the proof of the theorem of zeros which uses the dichotomy)… but how can we divide the bisector line?

R19**: Fr:** then it is not the one of the zeros…it is of Weierstrass

R20**: Dan:** (he tries to draw the function) it does like this…and then it will go to B

R21**: Fr:** we can do like this and then going down straight to B

R22**: Dan:** f(a)>x   f(b), namely one of the possible…I mean whatever could be…*a* could not do…and yes because *b* at most is here…that means that this point must stay always over x

R23**: Fr:** do you want to say that if *a* is over, *b* must stay below, and vice versa?

R24**: Dan:** exactly, otherwise it doesn't take everything, but the worst case if *a* is here to take all the values it should do like this and it would not continuous anymore…

R25**: Fr:** why like this?

R26**: Dan:** because in the same point it takes infinite values…at least I think…wait a second…if $a>0$ and $b<0$ the function must intersect the axis therefore the issue is always the same…

R27**: Fr:** oh yes…instead of the x-axis we have a line

R28**: Dan:** the bisector line…

R29**: Fr:** ah but then it is done…considering $a>0$, namely, *a* is greater than…

R30**: Dan:** *a>x….f(a)…*

R31**: Fr:** ah yes… $f(a)>x \quad f(b)<x$ and we know that must be this because it must take all the values…

R32: **Dan:** this should be a fixed point *f(x)=…let's do f(x)=c*

R33: **Fr:** and this must be like this because it must take all the values

R34: **Dan:** or like that or the contrary…no no

R35: **Fr:** no the contrary not because it can't go under this, it must a value greater than x and this one less than x.

R36: **Dan:** ok divide it in two (he traces a line to divide the bisector line in two…in one part he draws the axes and he tries to reproduce the graphical proof of the theorem of zeros)…then we would say if this one is here and that one is there then we have an intersection for sure, but I don't remember…

R37: **Fr:** this is a proof because we said…if there is a point over and a point below and if the function is continuous…there must be an intersection…there exists a point c such that f(c)…the theorem of zeros said f(c) = 0…if it is the theorem of Weierstrass…

R38: **I:** Daniele is not convinced…then Francesca repeats…

R39: **Dan:** well…but we have to prove that the fixed point exists…

R40: **Fr:** yes but if we say…this is our condition in order all the values to be taken…it takes all the values only if one is over and the other one is under…for the theorem of Weierstrass there exists a point belonging to it…for sure because it said: he put the line in this way but it is the same thing and it said if a point is over and the other one is under there exists a point on the line because the function is continuous…therefore it is the same thing if the line is the bisector line…

R41: **I:** Daniele thinks…

R42: **Dan:** (talking to me) *is it enough in this way?…I mean, if it is a proof that can be accepted or not (Daniele explains the proof)…by the moment that it must take all the values of the Image, a > x   b < x…(he corrects himself)   f(a) > x f(b) < x*

R43: **I:** what is x?

R44: **Dan**: x is the bisector line, otherwise if b were here it could not take all the values of the Image because the function could not do like this (and he traces a vertical line)…

R45: **I:** I mean…because *f(b)* must stay under the bisector line?

R46: **Fr:** no no that's true…not necessarily *f(b),* but there is a point below the bisector line therefore if there is a point over the bisector line and one below not necessarily *f(a)* and *f(b)* if we call *0 "a"* e 1 *"b"*, then for the theorem of Weierstrass intersects the line for sure

R47**: I:** why the theorem of Weierstrass?

R48**: Fr:** because we studied the theorem of zeros that says that if the function is continuous and there is a point where the function is over the x-axis and one which is under then there exists an x such that $f(x)=0$, same thing for Weierstrass…if I shift the line…perhaps it is not the theorem of Weierstrass…

R49**: I:** But the theorem of Weierstrass is that one which says (and I state it)

R50**: Fr:** ah no…anyway, the theorem of the zeros shifted up…for example this is the line x = 2 there is necessarily a point f(x) = 2 and therefore the same thing if we take the bisector line as the line…there is a point that is over…one that is under…there must exist necessarily a point that lies on the bisector line

R51**: I:** Why?

R52: **Fr**: *because the function is continuous*

R53**: I:** Then prove exactly this…if the function is continuous it intersects the bisector line…how would you prove that if the function satisfies the conditions then there is a point of intersection with the bisector line

R54**: Dan:** If I divide the bisector line in several intervals…

R55**: I:** how?

R56**: Dan:** if f in the new interval (a,α) (but it takes it on the bisector line)…if *f(a)*…

R57**: I:** but the interval on the line how do you take it? How do you define it?

R58**: Dan:** I would divide the segment…this is a known distance, isn't this? It is the diagonal of the square that is $\sqrt{2}$ …and therefore I don't know…

R59**: I:**Did you understand what I want to say?…the idea is interesting, but you have to tell me how you divide the line (the graphic aid is very important)

R60**: Dan:** if this is α for example…if I divide this which can be considered a segment…into two equal parts… $f(\alpha)$ is still > α…it means that there could be an intersection with the bisector line

R61**: Dan:** I mean I continue dividing it…

Appendix C: Scanner of the protocols

The following pages present the scanner of the protocols produced by the students.

The first protocol has been produced by Marco and Matteo in the solution of the problem about the fixed point (p.244).

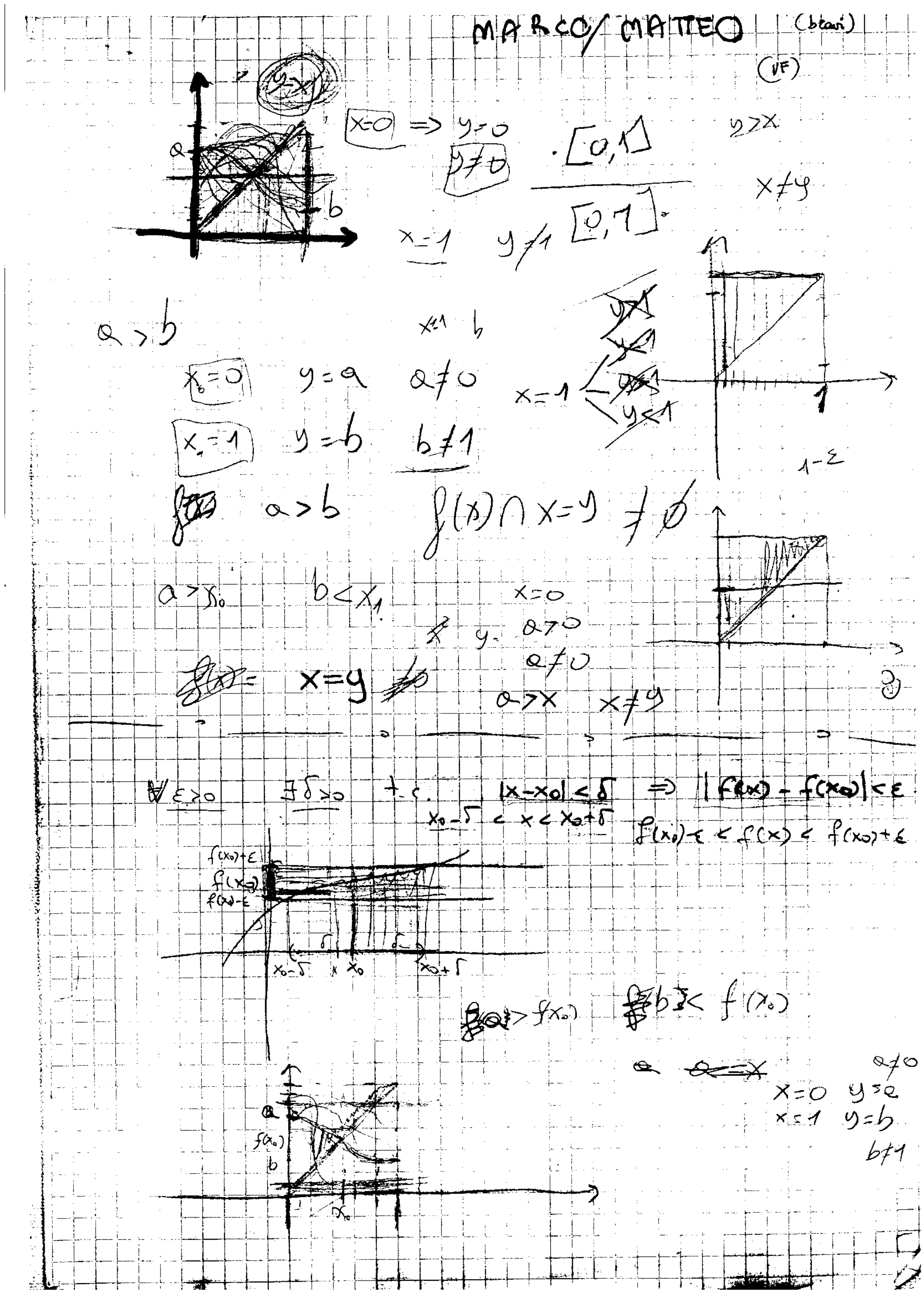
The second protocol shows the work done by Alice and Roberta during their attempt to solve the fixed point problem (p. 245).

The third protocol concerns again the solution about the fixed point and it has been produced by Francesca and Daniele (p. 246).

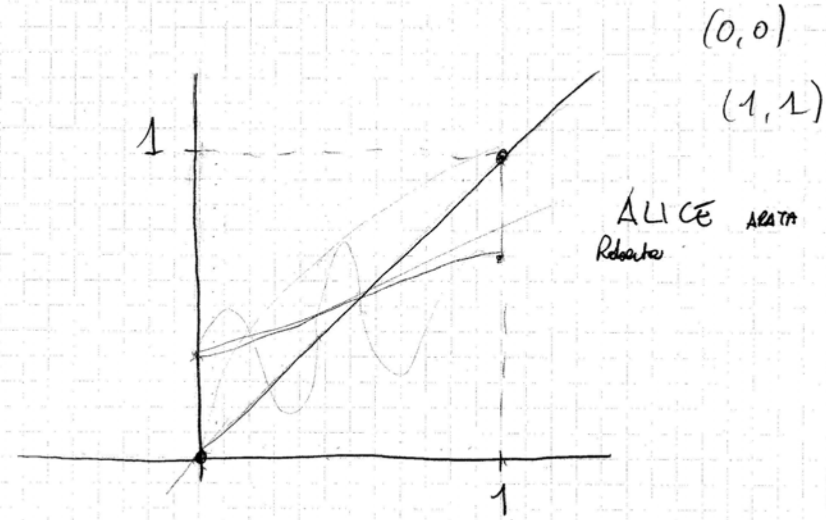
The fourth protocol regards the problem about the limit and it is Betta and Daniele’s work (p. 247).

The fifth protocol has been produced by Alice and Marco and it shows their attempt in the solution of the problem regarding the limit (p. 248-250)

The last protocol is again about the limit problem and it is the result of Francesca and Serena’s attempts (p.251-252)



Marco and Matteo’s protocol



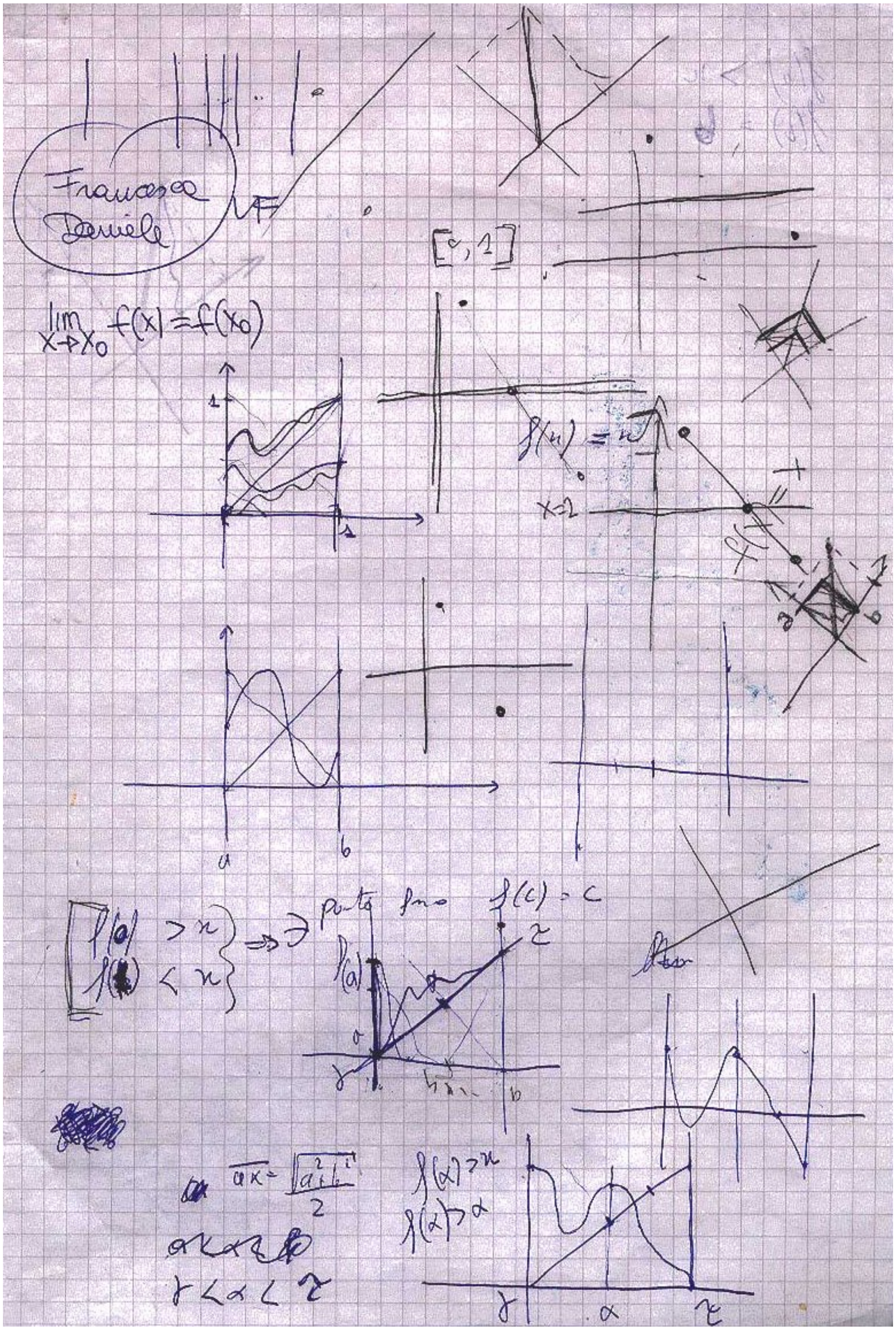
Per ipotesi deve esistere un punto di ascisse = 0
e di ordinate comprese tra 0 e 1
$P\ (0,\ 0 \leq y \leq 1)$ perchè $Dom = [0,1]$ e
$cod = [0,1]$

Deve anche esistere un punto di ascisse 1
e ordinate compresa tra 0 e 1:
$P_1\ (1,\ 0 \leq y \leq 1)$

con $P(0,0)$ e ~~$P_1(1,1)$~~

se $f(x)$ passa per $P(0,0)$, un
punto fisso è P ; potrebbero esistere anche
altri punti fissi nel caso in cui la funzione
attraversi la bisettrice
Analogamente per se la funzione passa per il punto
$P(1,1)$

Alice and Roberta's protocol

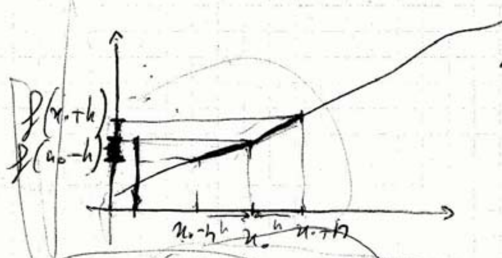


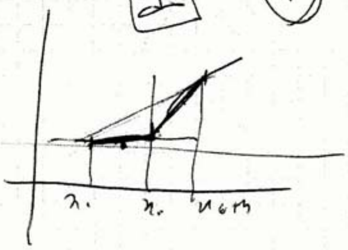
Francesca and Daniele's protocol

ACCO / BETTA

ESERCIZIO 1

1





$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

$$2\,\frac{f(x_0+h)-f(x_0)}{h} \neq \frac{f(x_0+h)-f(x_0-h)}{2h}$$

$$2f(x_0+h)-2f(x_0) = f(x_0+h)-f(x_0-h)$$

$$f(x_0+h)+f(x_0-h) = 2f(x_0)$$

$$f(x_0) = \frac{f(x_0+h)-f(x_0-h)}{2} + f(x_0+h)$$

$$\frac{f(x_0+h) - \dfrac{f(x_0+h)-f(x_0-h)+2f(x_0-h)}{2}}{h}$$

$$\left(f(x_0+h) - \frac{f(x_0+h)+f(x_0-h)}{2}\right)\cdot\frac{1}{h}$$

$2f(x_0+h) - f(x_0$

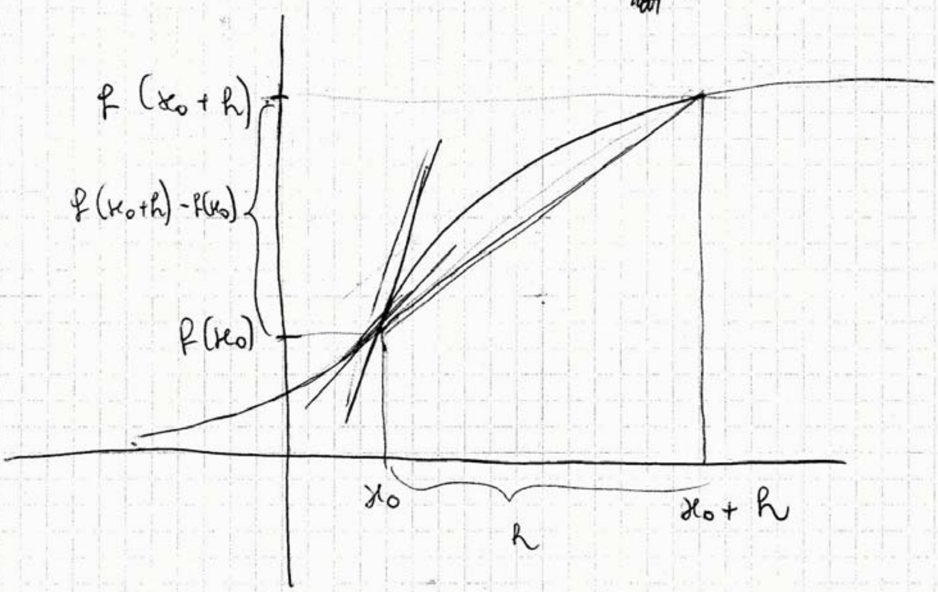
$$\frac{f(x_0+h)+f(x_0-h)}{2h}$$

Daniele and Betta's protocol

Es 1

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

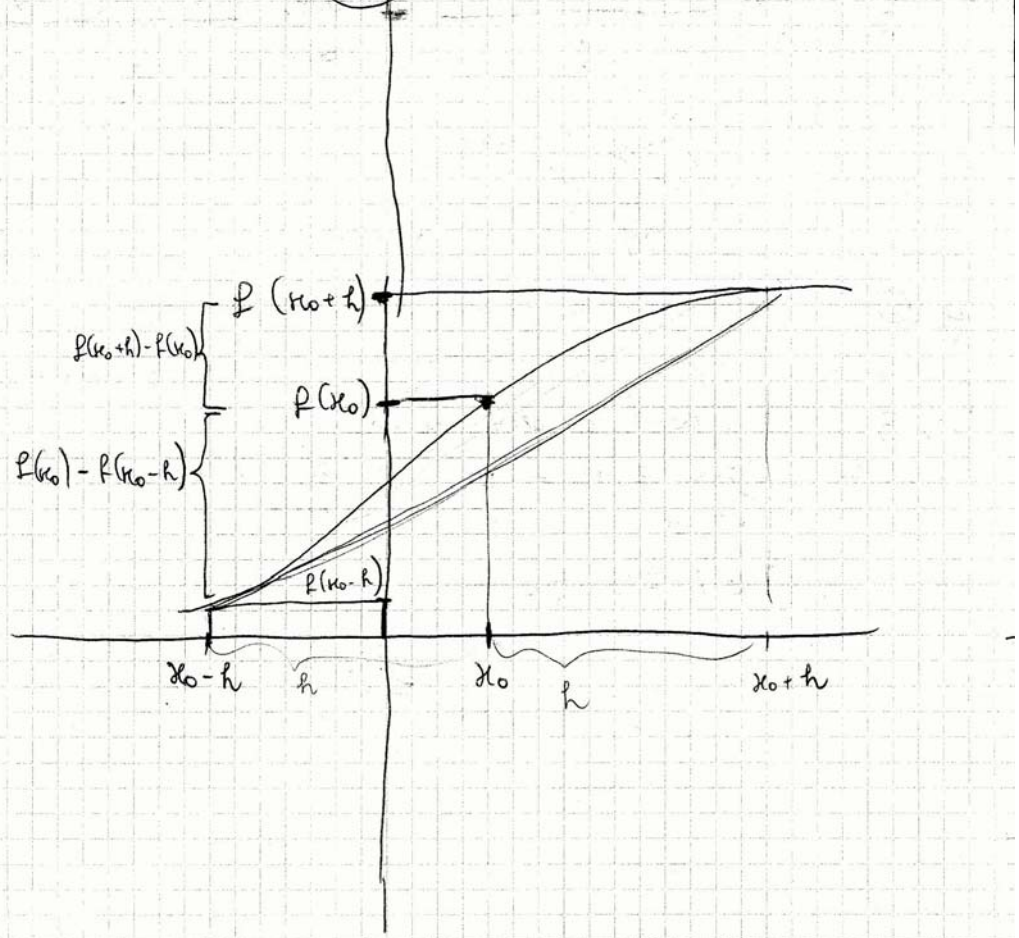
rapporto increm: $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$



PAG 2

Alice and Marco’s protocol (part 1)

continuazione 1



$\cancel{f(x_0-h)=f(x_0+h)}$

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}=\frac{1}{2}\lim_{h\to 0}\frac{f(x_0+h)-f(x_0+h)}{h}=$$

$$=\frac{1}{2}\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)+f(x_0)-f(x_0-h)}{h}$$

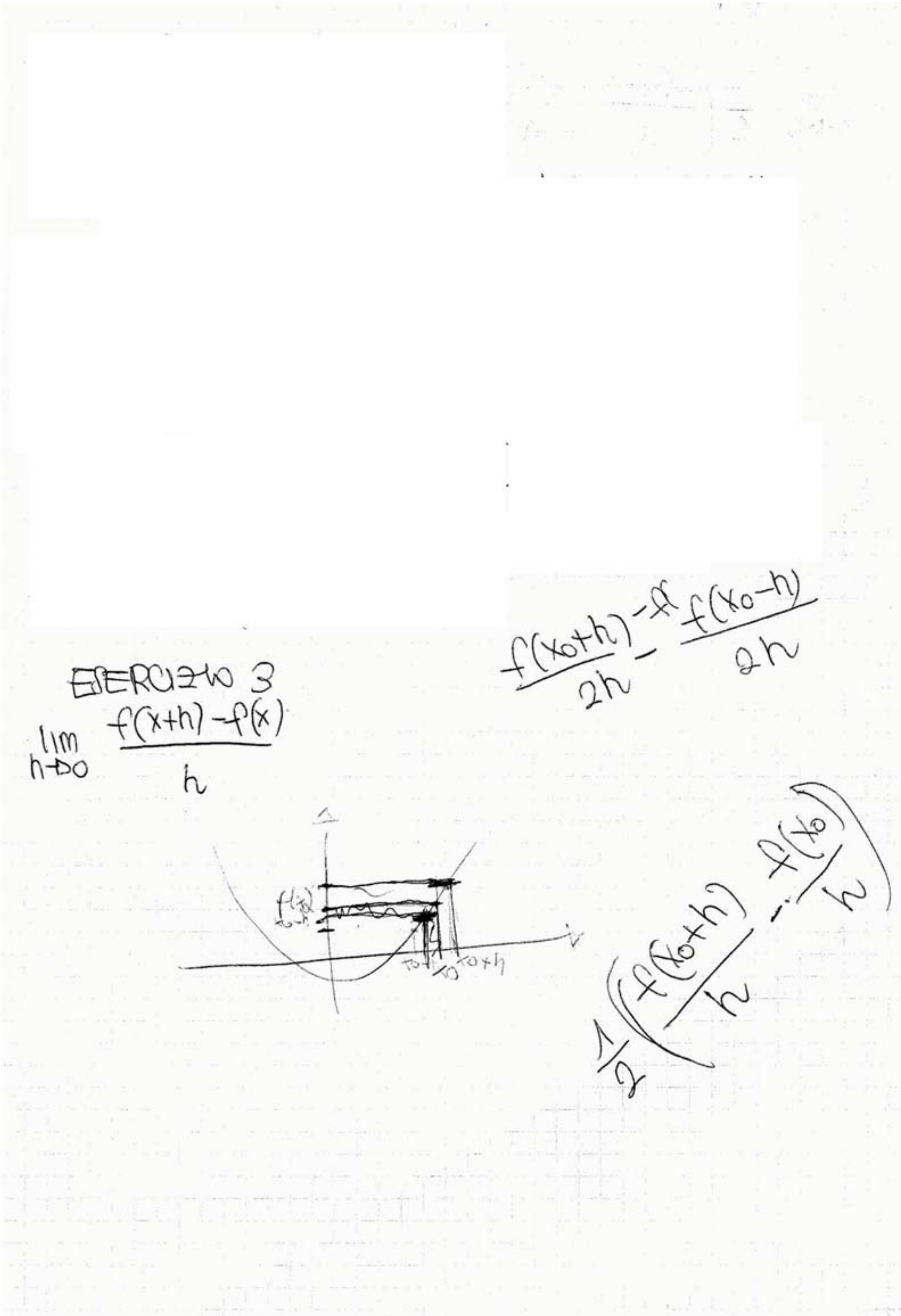
PAG 3

Alice and Marco’s protocol (part 2)

$$= \frac{1}{2}\left(\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} + \lim_{h \to 0} \frac{f(x_0-h) - f(x_0)}{h} \right)$$

$= 0$

Alice and Marco’s protocol (part 3)



Francesca and Serena’s protocol (part 1)

$$\lim_{h \to 0} \frac{1}{2}\left(\frac{f(x_0+h)}{h} - \frac{f(x_0)}{h}\right) + \frac{1}{2}\left(\frac{f(x_0)}{h} - \frac{f(x_0-h)}{h}\right)$$

$$\frac{1}{2} f'(x_0) + \frac{1}{2}\left(\frac{f(x_0-h) - f(x_0)}{-h}\right) = f'(x_0)$$

$$\frac{f(x_0-h) - f(x_0)}{-h}$$

Francesca and Serena’s protocol (part 2)

VITA

VITA

Elisabetta Ferrando

Academic Background

1996	MSc. in Mathematics, University of Genova, Italy
1997 - 1998	Specializing Course in Mathematics Education, “Analisi e Probabilità: aspetti teorici, storici, epistemologici e didattici”, University of Pavia, Italy

Professional Experiences

1998	Teaching Assistant, Engineering Department, University of Genova, Italy
1999 (Fall Semester)	Teaching Assistant (Math151, Calculus), Mathematics Department, Purdue University, Indiana, USA
2000 (Spring Semester)	Teaching Assistant (Math 132), Mathematics Department, Purdue University, Indiana, USA
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2002-2004	Research Assistant, Engineering Department, University of Genova, Italy
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Publications

Ferrando, E. (1998), “A Multidisciplinary Approach to the Interpretation of some of the Difficulties in Learning Mathematical Analysis”, *Proceedings of CIAEM – 50,* Neuchatel, vol.1 pp. 308-312.

Ferrando, E. (2000), “The Relevance of Peircean Theory of Abduction to the Developments of Students’ Conception of Proof (with a particular attention to proof in calculus)., *Proceedings of the Semiotics Society of America “semiotics 2000”,* West Lafayette, USA, chapter X pp. 1-16.

Papers Presented at Conferences

Il Continuo da un punto di vista epistemologico, e cognitivo. Presented at the University of Torino, May 23-24, 2001

Storia della dimostrazione e studio sul concetto di dimostrazione per studenti del primo anno di università, Presented at School of Advanced Mathematics, June 2003, Cogne, Italy

Roles of the Languages in Mathematics, Presented at School of Advanced Mathematics, June 2004, Cogne, Italy

La Congettura e la fase di evidenziazione nel processo dimostrativo. Alcuni strumenti cognitive per interpretare processi e difficoltà nell’ambito dell”analisi Matematica, Presented at the 2004 Matematica e Scuola: facciamo il punto. Milan, 13-14-15 October 2004. Centro Scolastico Gallaratese _I.T.T. Artemisia Gentileschi

Current Research Interests

The role of creative processes in mathematical proving processes

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